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Bruno de Finetti's encounter with martingales

Eugenio REGAZZINI[1]

and

Fabio SPIZZICHINO,[2]

Abstract: An account of the first encounter, in 1938, of de Finetti with the notion of martingale is given. The reasons, of an actuarial nature, which led him to deal with the subject are explained, along with a description of the ensuing original contributions to the specific field of martingales. The value of some of his conclusions is, then, discussed in the light of later development of the theory. The note is completed by a few remarks about an interpretative problem concerning the connection between the risk aversion criterion and the (actuarial) criterion based on the riskiness level.

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[1]Università degli Studi di Pavia. eugenio.regazzini@unipv.it

[2]Università degli Studi "La Sapienza", Roma. Fabio.Spizzichino@uniroma1.it

1 Introduction

The present note deals with an episode which can naturally be situated in the history of martingales: the encounter of Bruno de Finetti (1906-1985) with martingales. This episode passed unnoticed in the June 2009 issue of the *Journal Electronique d'Histoire des Probabilités et de la Statistique*, entirely devoted to the historical aspects of the evolution of the theory of martingales. We think however that de Finetti's use of the concept of martingale in [13] deserves some attention within a reconstruction of the history of the theory. The present survey can, then, be seen as a complement to the impressive source of information contained in the mentioned issue of the *Journal Electronique*. In fact, this is not the first time that this piece of de Finetti's work is mentioned in historical accounts. See [6], [8], [28] and pages XXXV–XXXVI of [17]. De Finetti came to the concept of martingale indirectly, in the frame of a study on the Filip Lundberg (1876-1965) *collective risk theory*, with the aim of providing an intuitively significant interpretation of the parameter that appeared in the well-known inequality concerning the ruin probability, nowadays referred to as *Cramér-Lundberg inequality*. See (68) in [7] and, for a modern deduction in the spirit of de Finetti's viewpoint, cf. Exercise 10.2.2 in [21]. De Finetti's paper [13], published in the first half of 1939, is taken into consideration in this note since it contains some statements about martingale-like properties, stopped processes and optional stopping. Note that, as it can be inferred from the combination of footnote 23) of [14] with that on page 1 of the same paper, the content of [13] had actually been included, almost entirely, in the first draft of [14], which was submitted before the end of 1938. So, de Finetti's statements concerning martingales and optional stopping go back to 1938. The sole authors who, out of historical evocation, have mentioned [13] were Lester E. Dubins (1920-2010) and Leonard J. Savage (1917-1971). See the many quotations made in [20].

This short note aims at briefly discussing some aspects of the de Finetti contribution and is organized as follows. Section 2 contains a survey of de Finetti's statements concerning the martingale property and a suitable exponential transformation he traces back to Abraham De Moivre (1667-1754). In the third section we present a short account about de Finetti's *coherence principle* and the ensuing theory of conditional probabilities. Therein we also hint, in a way which is consistent with such principle, at the argument that one can use to prove the propositions recalled in Section 2. In fact, in de

Finetti's original paper proofs are omitted. The fourth section is devoted to an analysis of the entailments, of an actuarial nature, of de Finettis' results and, in particular, of the exponential transformation recalled in Section 2. A reconsideration of the meaning of such transformation is finally outlined in the Section 5, according to the probabilistic interpretation of utility function, presented by de Finetti in Section 9 of [15].

2 Gamblers' ruin and de Finetti's use of martingale condition

De Finetti considers the sequence $(X_n)_{n\geq 1}$ of the random yearly gains of an insurance company, whose initial capital, at time 0, is G'. He assumes that the X_n's spring from a sequence of games against an opponent with initial capital G'' at his disposal, and that the game continues until one player is ruined. The time of the ruin of the company is then

$$\tau' := \inf\{n \geq 1 : Y_n \leq -G',\ -G' < Y_k < G'' \text{ for every } k \leq n-1\}$$

with

$$Y_0 := 0, \qquad Y_n := X_1 + ... + X_n \quad \text{if } n = 1, 2, ... \ .$$

Likewise,

$$\tau'' := \inf\{n \geq 1 : Y_n \geq G'',\ -G' < Y_k < G'' \text{ for every } k \leq n-1\}$$

represents the time of the ruin of the opponent. Hence, the play stops at $\tau := \tau' \wedge \tau''$.

De Finetti defines P_n^* to be the probability of the event "no ruin by time n", i.e. $\{-G' < Y_k < G''$, for every $k = 1, ..., n\}$, P_n' to be the probability of the event "ruin of the company by time n", i.e. $\{\tau' \leq n\}$, and P_n'' the probability of "ruin of the company by time n", i.e. $\{\tau'' \leq n\}$. Then, $P_n^* + P_n' + P_n'' = 1$. At this stage, he formulates a first hypothesis, i.e.

(a) $\lim_{n\to\infty} P_n^* = 0$,

under which, $P' := \lim_{n\to\infty} P_n'$ and $P'' := \lim_{n\to\infty} P_n''$ turn out to be the probabilities of (ultimate) ruin of the company and of the opponent, respectively, and τ will be finite with probability one. Clearly, $P' + P'' = 1$ and, in the case of fair games, as taken into consideration by the classical treatments, one gets

$$P''G'' - P'G' = 0. \tag{1}$$

Setting aside particular conditions under which (1) is shown to hold true, de Finetti points out that the necessary and sufficient conditions for (1) to be valid reduce to the following:

(b) The play must be fair until one player is ruined, i.e.

$$\mathbb{E}\left(Y_\tau\right) = 0.$$

(c) At time τ, the possibility of outstanding loss margins is precluded, that is

$$Y_{\tau'} + G' = Y_{\tau''} - G^{''} = 0.$$

In the classical treatments, (a) and (b) are obtained as a consequence of the fact that the X_n are thought of as independent random numbers in $\{-1, 1\}$, with the same symmetric distribution. Then, (c) is also met by assuming that both G' and G'' are strictly positive integers. But, to deal with the case of an insurance company, the above assumption of independence turns out to be inadmissible: suffice it to mention the risk "death of a well-specified individual, considered over a sequence of years". Likewise, the condition of fairness of the games clashes with the practice of calculating premiums that are suitably loaded in order to provide the company with a margin for fluctuations.

As to (c), its possible removal, provided that (a) and (b) continue to be valid, will entail a simple modification of (1) into

$$\left(G'' + \Delta''\right) P'' + \left(-G' + \Delta'\right) P' = 0, \tag{2}$$

where $\Delta' := E\left(Y_{\tau'} + G' | \tau' < +\infty\right)$ and $\Delta'' := E\left(Y_{\tau''} - G'' | \tau'' < +\infty\right)$ denote the conditional probabilities of loss outstanding margins given the ruin of the company and of the opponent, respectively. Hence

$$P' = \frac{G'' + \Delta''}{G' + G'' + \Delta'' - \Delta'}.$$

Much more interesting, at least from the point of view of the theory of martingales, is how de Finetti discusses the way of deriving (b) from some sensible conditions, more general than stochastic independence. At this point, the present survey splits into two subsections dealing with fair and unfair gambles, respectively.

2.1 The case of fair gambles

In order to assure the validity of (b), de Finetti proposes to assume that *each game turns out to be fair conditionally on any hypothesis about the previous games*, a condition that he denotes with (b"). In formulas,

(b") $\mathbb{E}(X_1) = 0$, $\mathbb{E}(X_{n+1}|X_1 = x_1, ..., X_n = x_n) = 0$, for every $n \geq 1$ and for every logically possible event $\{X_1 = x_1, ..., X_n = x_n\}$.

In de Finetti's language the locution "logically possible", referred to an event, stands for "different from the impossible event" (see the detailed discussion presented in Chapter 2 of [16]). More information on de Finetti's conception of conditional probabilities and expectations is given in Section 3.

For the sake of completeness, it should be recalled that he mentions also a further condition, stronger than (b) and weaker than (b"). It is

(b') $\mathbb{E}(X_1) = 0$, $\mathbb{E}(X_j| - G' < Y_k < G'', \text{ for } k = 1, ..., j-1) = 0$, for every $j \geq 2$.

De Finetti confines himself to saying that (b') is equivalent to asserting that the game continues to be fair until $\tau \wedge m$ for every $m = 1, 2...$: $\mathbb{E}(X_{\tau\wedge m}) = 0$, $m = 1, 2....$ But he claims that the sole condition (b") will be considered in the remaining part of his work.

It is clear that (b") can be thought of as a martingale-like condition: when it is in force, $(X_n)_{n\geq1}$ is a *martingale difference sequence*, and $(Y_n)_{n\geq1}$ is a *martingale*. Besides, (b") is the same as the "Lévy martingale condition" ($\mathfrak{C}$), deeply discussed in [26]. Paul Lévy (1886-1971) had used ($\mathfrak{C}$) in [23], a paper of 1929 and, much more profitably, in Chapter VIII of his treatise [24] of 1937, to study the limiting behavior of sums of non-independent random variables. It is sure that, in 1938, de Finetti was acquainted with Lévy's treatise, as shown by the precise quotations made, for example, on pages 18 and 39-40 of [11] and on pages 31 and 56-57 of [12]. In spite of this, in [13] there is no quotation of Lévy's book, a fact whose real reason is difficult to discover. De Finetti was used to carefully mention references, including those having little bearing on the development of his own research. So, one could be led to think that he had not read Chapter VIII of Lévy's book carefully. In any case, the use that he makes of condition (b") is radically different from the one that Lévy makes of condition ($\mathfrak{C}$). In fact, according to the main aim of the paper under discussion, de Finetti's motivation for condition (b") is the advisability of using such condition in place of the aforesaid (necessary) condition (b). It is just the desire to justify this substitution that guides him to state on page 45 of [13] that (b"), apart from being sufficient for the validity of (b),

> *Is necessary and sufficient in order that fairness of the play is preserved until the play is interrupted, and interruption is allowed to occur either because of the ruin of one of the two players or because of the unilateral decision to drop out the game by the company, by the opponent or by anyone of them.*

In view of this statement, de Finetti can be actually considered as a forerunner of a partial version of the so-called *optional stopping theorem*, that was proved independently and in a more general setting, but fifteen years later, by Joseph Doob (1910-2004). See [19], Section 2 of Chapter VII and the corresponding Appendix on page 630.

2.2 The case of unfair gambles

De Finetti extended the previous way of reasoning to the case of unfair games, of paramount importance for the actuarial nature of the paper. With a view to this, he reformulated, in more abstract and general terms, an idea by De Moivre he had learned from Chapter VI of the monograph [4] by Joseph Bertrand (1822-1900). The method requires extra-conditions such as

(d) There is $\delta > 0$ such that $\mathbb{E}\left(e^{\alpha X_n}\right)$ is finite for every α in $(-\delta, \delta)$ and for every n. Moreover, there is a constant $\alpha_0 \neq 0$ in $(-\delta, \delta)$ such that $\mathbb{E}\left(e^{\alpha_0 X_1} - 1\right) = 0$ along with

$$\mathbb{E}(e^{\alpha_0 X_{n+1}} - 1 | X_1 = x_1, ..., X_n = x_n) = 0$$

for every $n \geq 1$ and every logically possible event $\{X_1 = x_1, ..., X_n = x_n\}$[3].

[3] The existence of such an α_0 different from zero is strictly linked with the assumption that the fairness assumption is here removed. De Finetti easily shows that the sign of α_0 is opposite to that of $\mathbb{E}(X_1)$.

De Finetti argues that, under (a) and (d), if the barriers $-G'$ and G'' are transformed into $\left(e^{-\alpha_0 G'} - 1\right)$ and $\left(e^{\alpha_0 G''} - 1\right)$ respectively, and if the cumulative annual gains Y_n are transformed into $\left(e^{\alpha_0 Y_n} - 1\right)$ for every n, then the ruin probabilities, along with τ, do not change, with respect to the original process $(Y_n)_{n\geq 1}$. He argues also that the sequence $\left(e^{\alpha_0 Y_n} - 1\right)_{n\geq 1}$ satisfies the martingale condition and, then, $\mathbb{E}\left(e^{\alpha_0 Y_\tau} - 1\right) = 0$. This entails

$$1 = P' \cdot \left[e^{-\alpha_0 G'} + \mathbb{E}\left(e^{\alpha_0 Y_{\tau'}} - e^{\alpha_0 G'} | \tau' < +\infty\right)\right] +$$
$$P'' \cdot \left[e^{\alpha_0 G''} + \mathbb{E}\left(e^{\alpha_0 Y_\tau} - e^{\alpha_0 G''} | \tau'' < +\infty\right)\right]$$

with $P' + P'' = 1$. Then,

$$P' = e^{\alpha_0 G'} \frac{e^{\alpha_0 (G''+D'')} - 1}{e^{\alpha_0 (G'+G''+D'')} - e^{\alpha_0 D'}} \tag{3}$$

holds true with D' and D'' defined as conditional exponential means of the outstanding loss margins, i.e.

$$D' := \frac{1}{\alpha_0} \log \mathbb{E}\left(e^{\alpha_0 (Y_{\tau'} + G')} | \tau' < +\infty\right)$$

and

$$D'' := \frac{1}{\alpha_0} \log \mathbb{E}\left(e^{\alpha_0 (Y_{\tau''} - G'')} | \tau'' < +\infty\right).$$

It is worth stressing that the aforesaid argument, which has led to (3), shows that de Finetti was able to carry his method, based on the martingale condition, even beyond the narrow field of the martingale difference sequences. It should also be noticed that, by means of the optional stopping argument, he had anticipated the fundamental identity of sequential analysis generally attributed to Abraham Wald (1902-1950). See [32] and Section 10 of Chapter VII in [19]. See also the appreciation of de Finetti's work made by Dubins and Savage apropos of their *exponential houses* in [20].

3 Hints on de Finetti's theory of conditional distributions

The use of ordinary language to explain, in Section 2, the main results concerning the martingale condition, reflects the style of de Finetti's original paper. A further feature of the latter is the lack of any hint for proving the aforesaid statements, which refer to the essential properties of each of the conditions (b), (b'), (b"), and to the implications (b") $\rightarrow$ (b'), (b') $\rightarrow$ (b). Combining this circumstance with the fact that de Finetti's conception, and consequent mathematical theory of conditional expectations and conditional probabilities, differs from that of Kolmogorov – within which the standard theory of martingales has been developed – induces one to ponder on de Finetti's possible reasoning behind the deduction of some of the results stated in [13]. In point of fact, such results substantially coincide with those that one commonly proves within the Kolmogorov theory. So, the problem is that of showing how these statements can be checked within a suitably modernized version of de Finetti's language. This is the aim of the present section. In order to make things easier, events will be thought of as subsets of a set Ω of elementary cases, and random variables will be identified with functions on Ω. According to de Finetti, no measurability requirement will be considered as an integral part of the definition of random element in general. The same symbol that designates an event also designates its indicator. Given any random variable X and any event $H \neq \emptyset$, the restriction of X to H, denoted by $X|H$, is said to be the *conditional random variable X given H*. So, in de Finetti's theory, conditioning is defined with respect to single events (hypotheses) rather than with respect to classes of events. This explains the notation used in Section 2 to denote expectation of conditional real-valued random variables (random numbers, r.n.'s, for short). When X is the indicator of some event, then $X|H$ is said to be the *conditional event* given H. Conditional random variables given the certain event Ω reduce of course to (unconditional) random variables. Hence the following treatment - which heavily relies on [3] - covers any class of bounded r.n.'s and events, even if it explicitly mentions classes of conditional bounded r.n.'s only.

Given a class $\mathcal{C}$ of bounded conditional r.n.'s, $P : \mathcal{C} \rightarrow \mathbb{R}$ is said to be a *prevision* on $\mathcal{C}$ if it meets the *coherence principle*. De Finetti's work in probability has been informed by this principle from the year 1930. See [30] for more information. In particular, if $\mathcal{C}$ is a class of conditional events, then a prevision P is said to be a *probability* on $\mathcal{C}$. In de Finetti's theory of probability, the concept of prevision is the counterpart of that of expectation in the standard theory and we shall use the two terms as synonyms. As far as the coherence principle is concerned, we can provide the reader with the following explanation. Suppose that after assigning P on $\mathcal{C}$, I am committed to accepting any bet whatsoever on each element of $\mathcal{C}$, with arbitrary (positive or negative) stakes, on the understanding that any bet on $X|H$ is called off if H does not occur. In this framework, $P(X|H)$ represents the unit price of every bet on $X\ |H$. Then, what I gain from a combination of bets on $X_1|H_1, ..., X_n|H_n$, with stakes $s_1, ..., s_n$ respectively, is given by

$$G = G(X_k|H_k, s_k : k = 1, ..., n) = [\sum_{k=1}^{n} s_k H_k \{P(X_k|H_k) - X_k\}]|H_0$$

with $H_0 = \bigcup_{k=1}^{n} H_k$. Then P is said to be *coherent* if it does not allow *anybody* to make Dutch Book against me, i.e.

$$\inf G \leq 0 \leq \sup G$$

holds true for every choice of n in $\mathbb{N}$, $\{X_1|H_1, ..., X_n|H_n\} \in \mathcal{C}$, and $(s_1, ..., s_n)$ in $\mathbb{R}^n$.

The above definition of prevision makes sense because, for any given $\mathcal{C}$, there exists at least one prevision on $\mathcal{C}$. Such a statement is a direct consequence of the following extension result:

Let $\mathcal{C}$ and $\mathcal{C}'$ be classes of bounded conditional r.n.'s such that $\mathcal{C} \underset{\neq}{\subset} \mathcal{C}'$ and let P be a prevision on $\mathcal{C}$. Then there exists a prevision (which need not be unique) P' on $\mathcal{C}'$ for which

$$P'(X|H) = P(X|H) \text{ for any } X|H \in \mathcal{C}.$$

See [2], [3], [29] and references cited therein. In order to deal with problems of the same kind as those concerning condition (b") in Section 2, it is enough to consider $\mathcal{C}$ defined as follows

$$\mathcal{C} := \{X|H : X \in \mathcal{L}, H \in \mathcal{H}\} \tag{4}$$

where $\mathcal{H} = \bigcup_{n \geq 0} \Pi_n$, $(\Pi_n)_{n \geq 0}$ being a sequence of partitions of Ω such that: Π_{n+1} is a refinement of Π_n for every $n \geq 0$ and $\Pi_0 = \{\Omega\}$, $\mathcal{L}$ is any class of bounded r.n.'s such that $\mathcal{H} \subset \mathcal{L}$ and $XH \in \mathcal{L}$ whenever $X \in \mathcal{L}$ and $H \in \mathcal{H}$.

As proved in [2], previsions admit an interesting characterization when their support is defined as in (4):

Let $\mathcal{C}$ be defined according to (4). Then $P : \mathcal{C} \to \mathbb{R}$ is a prevision if and only P satisfies the following conditions:

(p_1) $P(\cdot|H)$ *is an expectation on* L *for every* $H \in H$;

(p_2) $\inf X|H \leq P(X|H) \leq \sup X|H$ *for every* $X|H \in C$;

(p_3) $P(XH_1|H_2) = P(X|H_1 \cap H_2)\, P(H_1|H_2)$ *for every* $X, H_1 \in L$ *and* $H_2 \in H$ *such that* $H_1 \subset \Omega, XH_1 \in L$, *and* $H_1 \cap H_2 \in H$.

If $\mathcal{L}$ is a class of events, then (p_1)-(p_3) can be restated in the following form:

(π_1) $P(\cdot|H)$ *is a probability on* L *for every* $H \in H$;

(π_2) $P(A|H) = 1$ *whenever* $H \in H$, $A \in L$ *and* $H \subset A$;

(π_3) $P(A \cap H_1|H_2) = P(A|H_1 \cap H_2)\, P(H_1|H_2)$ *provided that* A, H_1, *and* $A \cap H_1$ *belong to* L *and* $H_1 \cap H_2$, H_2 *are elements of* H.

Further characterizations can be stated under additional conditions on $\mathcal{L}$. They provide an exact picture of the mathematical nature of the concepts of probability and expectation, in de Finetti's sense, when their domain reduces to the domain of their Kolmogorov-type counterparts. These characterizations read:

Let $\mathcal{L}$ *be a linear space of bounded r.n.'s, including the constants. Then,* $Q : \mathcal{L} \to R$ *is a prevision if and only if it is a positive, linear functional on* $\mathcal{L}$ *such that* $Q(\Omega) = 1$.

Let $\mathcal{L}$ *be an algebra of events. In order that* $Q : \mathcal{L} \to R$ *be a probability, it is necessary and sufficient that it turns out to be a non-negative-valued, additive function such that* $Q(\Omega) = 1$.

These statements summarize, in a precise way, correspondences and differences between de Finetti's approach and Kolmogorov's. The differences become more marked when moving to conditional concepts. As a consequence of the above basic definitions and results one has that previsions can be assessed on *arbitrary* classes of conditional bounded r.v.'s. In particular, one can fix a conditional prevision without pre-assigning any unconditional probability law. In de Finetti's theory, consequently, conditional probabilities need not be evaluated as derivatives of finite measures with respect to probability measures. Moreover, they need not be continuous with respect to monotone sequences of sets. But, the main difference between the two approachs is in that the former explicitly considers conditional random variables and conditional events given a single event, which may even have probability zero. On the contrary the latter (see page 51 of the English translation of [22]) asserts that "the concept of conditional probability with regard to an isolated given hypothesis whose probability equals zero is inadmissible". Kolmogorov translates this statement into a *disintegrability condition*, that holds also in de Finetti's approach for finite partitions only, but need not hold for infinite partitions. See in this respect the classical de Finetti's paper [9] concerning the lack of conglomerability. On the other hand, the class of admissible previsions includes a subclass of conditional previsions which satisfy the usual conditions that, within Kolmogorov's theory, are requested in order to let a random variable be a conditional expectation. Such a subclass has been studied in depth and systematically used in [20]. These conditional previsions can be called "strategic", in view of their connection with the concept of strategy due to Dubins and Savage. It is sensible to assume that de Finetti rested on some primitive idea of "strategic" expectation to deal with the matter considered in [13]. In fact, it is possible to prove the results recalled in Section 2 of the present note, within the following framework.

Let $\mathcal{X}$ be a set ($\mathbb{R}$ in de Finetti's paper), and put $\Omega = \mathcal{X}^\infty$. Call *histories* the elements of Ω, and *partial histories* the elements of $\mathcal{X}^n$, for every n. Now, define Π_n as follows, for any $n \in \mathbb{N}$,

$$\Pi_n := \{\{(x_1, ..., x_n)\} \times \mathcal{X}^\infty : (x_1, ..., x_n) \in \mathcal{X}^n\}$$

that is a partition of cylinders $I(x_1, ..., x_n) := \{(x_1, ..., x_n) \times \mathcal{X}^\infty\}$ determined by the partial histories of length n. So, in view of (p_1)-(p_3) and (π_1)-(π_3), any prevision P on $\mathcal{C}$ can be characterized by means of the following four conditions:

(i) $P(\cdot)\,,\ P(\cdot|x_1, ..., x_n)$ *are previsions* on $\mathcal{L}$

(ii) $P(I(x_1, ..., x_n)|x_1, ..., x_n) = 1$

(iii) $P(X \cdot I(x_1, ..., x_k, , ..., x_n)|x_1, ..., x_k) =$
$= P(X|x_1, ..., x_n)\, P(I(x_1, ..., x_n)|x_1, ..., x_k)$

(iv) $P(X \cdot I(x_1, ..., x_n)) = P(X|x_1, ..., x_n)\, P(I(x_1, ..., x_n))$

for every X in $\mathcal{L}$, $(x_1, ..., x_n)$ in $\mathcal{X}^n$, and for every k in $\{1, ..., n\}$.

In Dubins and Savage's terminology, a *strategy* is a sequence $\sigma = (\sigma_0, \sigma_1, ...)$ in which σ_0 is a probability on $\mathcal{P}(\mathcal{X})$, the power set of $\mathcal{X}$, and, for every n in $\mathbb{N}$, σ_n is a function on $\mathcal{X}^n$ which associates a probability on $\mathcal{P}(\mathcal{X})$, denoted by $\sigma_n(x_1, ..., x_n)$, to every partial history $(x_1, ..., x_n)$. If P satisfies (i)-(iv) and $\mathcal{L}$ includes

$$\mathcal{B} := \{B \times \mathcal{X}^\infty, \mathcal{X}^n \times B \times \mathcal{X}^\infty : B \subset \mathcal{X}, n \in \mathbb{N}\},$$

then the strategy with

$$\sigma_0(B) := P(B \times \mathcal{X}^\infty), \sigma_n(x_1, ..., x_n)(B) := P(\mathcal{X}^n \times B \times \mathcal{X}^\infty | x_1, ..., x_n)$$

is said to be *the strategy induced by* P. Moreover, P is said to be *strategic* if

$$P(X) = \int P(X|x)\sigma_0(dx)$$

$$P(X|x_1, ..., x_n) = \int P(X|x_1, ..., x_n, x)\sigma_n(x_1, ..., x_n)(dx). \tag{5}$$

Conditions (5), combined with (ii), looks like the usual condition requested for conditional expectations in the Kolmogorov theory. In [3] it is shown that *if* $\mathcal{L} \supset \mathcal{B}$, *all elements of* $\mathcal{L}$ *are finitary functions, and* σ *is a strategy, then there exists a unique strategic prevision satisfying* (i)-(iv), *such that* σ *is the strategy induced by* P.

According to Dubins and Savage (Section 7 of Chapter 2 of [20]), X is *finitary* if, for every history h, there is an n such that $g(h) = g(h')$ for every history h' which agrees with h on the first n coordinates.

Actually, we have checked that de Finetti's assertions concerning the martingale condition can be verified for any strategic prevision generated by a strategy satisfying the martingale difference condition. As a matter of fact, it has also been necessary to make the following phrase precise: "... fairness is preserved until the play is interrupted ...". Of course, a concept of *stopping rule* is implicitly contained here. It is reasonable, in this respect, to conjecture that de Finetti was thinking of a stopping rule as an integer function t on $\mathcal{X}^\infty$ satisfying the following condition: if two histories h and h' share the same partial history of length $t(h)$, then $t(h') = t(h)$. Useful properties of stopping rules defined in this way are analyzed in [20]. In fact, de Finetti's anticipation of the optional stopping theorem can be verified by combining the above definition of stopping rule with the aforesaid concept of strategic prevision.

4 Connection with the mathematical theory of risk

This section aims at discussing the solution that, on the basis of the argument mentioned in the second section, de Finetti finds for the actuarial problem recalled in the first section. In doing so he displays an interpretation and a possible operational use of the parameter which appears in the Cramér-Lundberg inequality. To show this, first consider (3) and recall that $\alpha_0 < 0$, since $\mathbb{E}(X_1) > 0$ in the case of an insurance company. In view of this, combined with the definitions of D' and D'', it is plain to see that

$$P' \leq e^{\alpha_0 G'}, \tag{6}$$

which is the above-mentioned inequality. It is plain to link α_0 with a *riskiness index*. According to de Finetti's notation, α_0 is henceforth denoted as $-1/B$ (with $B > 0$) and B represents the desired index. See also the next section.

In addition to the assumptions (a) and (d) for the sequence $(X_n)_{n\geq 1}$ of the random yearly gains, de Finetti thinks of X_1 as a sum of independent and identically distributed random numbers, say $G_1, G_2, ...$, where G_j stands for the gain (in the first year) of the j-th policy, $j = 1, ..., N$. So, from

$$1 = \mathbb{E}(e^{\alpha_0 X_1}) = \left[\mathbb{E}(e^{\alpha_0 G_1})\right]^N,$$

one derives that *the index B of the whole of the affairs is the same as the riskiness index of each policy.* This simple statement is the key to the applicability of the probability of ultimate ruin to the company decision process, when – according to (d) – the affairs are maintained, through the years, at a constant level of riskiness.

De Finetti illustrates the meaning of this statement by means of the following example. Let $E_1, ..., E_N$ be stochastically independent events with constant probability p. It is assumed that the holder of the j-th policy receives the amount C (0, respectively) if E_j occurs (does not occur, respectively) on the payment of the premium $C\,(p+m)$, for some strictly positive m and every $j = 1, ..., N$. Therefore, m stands for a unit loading, and $(-1/B)$ must equal the non-zero solution of the equation (in α)

$$\begin{aligned} 1 = \mathbb{E}\left(e^{\alpha G_j}\right) = \\ p\exp\{\alpha C(p+m-1)\} + (1-p)\exp\{\alpha C(p+m)\}. \end{aligned} \tag{7}$$

A rough estimation of the root of this equation, gives

$$B \simeq C\frac{p\,(1-p)}{2m} \tag{8}$$

which indicates, for example, how to fix the unit loading when the company fixes the level B for the riskiness of the whole of the affairs and, at the same time, decides to retain the entire insurance amount C. On the other hand, (8) can be used to determine the insurance amount to be retained, when both B and m are fixed. Finally, it is not difficult to prove that a similar conclusion holds true in general. Namely, if σ^2 and m respectively represent the variance and the (strictly positive) expectation of the random gain associated with each policy, then $B \simeq C\sigma^2/2m$.

The arguments used by de Finetti's to get both the exact expression (3) and the bound (6) are actually characterized, from a conceptual point of view, by a remarkable simplicity. It is to be noticed in this respect that some of the most interesting aspects of this subject are somehow put in the shade by the analytically more elevated treatment presented by Lundberg. First, the latter does not allow determining the influence exerced by every single risk on the riskiness of the whole of the affairs, as it would be necessary in order to let the theory to accomplish its main task, that is the determination of the insurance amount to be retained. Furthermore, the Lundberg-Cramér formulation of the theory assumes stochastic independence among risks whether they are considered within the same business year or in different years. As to the results, their form is rather complicated and the analytical methods, used for their achievement, do not help to discover any intuitively expressive meaning for both the solutions (exact and approximate). As a matter of fact, as clearly proven in [7], $1/B$ is determined, under certain conditions, as the solution of an integral equation, that is somehow difficult to interpret. Hence, the argument proposed by de Finetti represents a definite improvement for this subject, an improvement that can be attributed to a valuable and pioneering application of a martingale condition.

5 Connection with the Bernoulli expected utility principle

De Finetti came back to the meaning of the coefficient α_0 in a paper devoted to a critical analysis of different standings on the foundations of utility theory, as arisen during a meeting on the subject, held in Paris from May 12 to May 17, 1952. See [15]. De Finetti calls *probabilistic* the behavior which agrees with the axioms proposed by John von Neumann (1903-1957) and Oskar Morgestern (1902-1977) in [31]. Hence, the probabilistic behavior resumes the classical expected utility principle which dates back to Daniel Bernoulli (1700-1782). According to it, each individual has a *utility function* u (a strictly increasing and continuous real-valued function on $\mathbb{R}$) so that, to any random gain G, with probability distribution function F, it is associated a non-random gain $\widehat{g}$, equivalent to G in the sense that

$$\int_{\mathbb{R}} u(x)\, dF(x) = u(\widehat{g}). \tag{9}$$

When $\widehat{g} = 0$, G can be seen as a "neutral" gain, since it does not alter, from the point of view of the expected utility, the initial economic situation of the individual taken into consideration. Coming back to the equation (9), consider the problem of the equivalent certain gain for the restriction F_0 of F to a small neighborhood $N_{x_0} := (x_0 - \varepsilon, x_0 + \varepsilon]$ of an arbitrary point x_0 in the support of F. More precisely,

$$F_0(x) := \begin{cases} 0 & \text{if } x < x_0 - \varepsilon \\ \frac{F(x)}{F(x_0+\varepsilon)-F(x_0-\varepsilon)} & \text{if } x_0 - \varepsilon \leq x < x_0 + \varepsilon \\ 1 & \text{if } x \geq x_0 + \varepsilon \end{cases}$$

Then the analogue of (9) is

$$\int_{N_{x_0}} u(x)\, dF_0(x) = u(\widehat{g}_0)$$

as an equation in the variable $\widehat{g}_0$, belonging to N_{x_0}. By an elementary estimation, which becomes better and better as ε decreases, the last equation roughly becomes

$$u(\widehat{g}_0) + \int_{N_{x_0}} (x - \widehat{g}_0)\, u'(\widehat{g}_0)\, dF_{x_0}(x) +$$

$$+\frac{1}{2} \int_{N_{x_0}} (x - \widehat{g}_0)^2\, u''(\widehat{g}_0)\, dF_0(x) \backsimeq u(\widehat{g}_0),$$

that is

$$\frac{\int_{N_{x_0}} (x - \widehat{g}_0)\, dF_0(x)}{\int_{N_{x_0}} (x - \widehat{g}_0)^2\, dF_0(x)} \backsimeq -\frac{u''(\widehat{g}_0)}{2u'(\widehat{g}_0)} \backsimeq -\frac{u''(x_0)}{2u'(x_0)}.$$

Thinking over the meaning of the left-hand-side, one discovers that $\left(-\frac{u''(x_0)}{2u'(x_0)}\right)$ has the meaning of a *local risk aversion index.* In fact, its value increases as the certain equivalent gain $\widehat{g}_0$ approaches the lower bound $(x_0 - \varepsilon)$ of the support of F_0, and/or the dispersion of F_0, around g_0, becomes smaller and smaller. De Finetti, just in [15], was the first to popularize this interpretation of $-\frac{u''(x)}{2u'(x)}$.

At the end of Section 9 of that very same paper, he mentions how some authors proposed a criterion based on the asymptotic risk theory as an alternative to the Bernoullian paradigm. Indeed, asymptotic risk theory would allow any economic agent to determine both the risk amounts to be retained and the equivalent certain gain in the form of a "loaded mathematical expectation". On the basis of the developments of actuarial interest explained in [13] (that we have reported in the previous section of this note), de Finetti argued as follows: the criterion of the "level of risk", that risk theory leads to follow, turns out to be exactly the same as the "probabilistic behavior" in the presence of utility functions of exponential form, i.e.

$$u(x) = k - e^{-2x/\lambda} \tag{10}$$

λ being some strictly positive constant. By the way one can check that (10) is the sole utility function with constant local risk aversion index $1/\lambda$. In order to verify de Finetti's statement, it is worth taking the same random gain that has been considered in Section 3 to illustrate the origin of the riskiness index B. The utility index of such a gain, as deduced from (10) is

$$pu\left(C\left(p+m-1\right)\right) + (1-p)\, u\left(C\left(p+m\right)\right) =$$

$$= k - e^{-\frac{2}{\lambda}C(p+m)}\left(pe^{-\frac{2}{\lambda}C} + 1 - p\right).$$

Then the certain equivalent $\widehat{g}$ must satisfy the equation

$$e^{-\frac{2}{\lambda}\widehat{G}} = e^{-\frac{2}{\lambda}C(p+m)}\left(pe^{-\frac{2}{\lambda}C} + 1 - p\right)$$

and, in particular, in order to get $\widehat{g} = 0$, i.e. the "neutrality" condition with respect to (10), C (the part of risk to be retained) and m (the appropriate correction for the mathematical expectation) must satisfy condition ((7))

with $\alpha = -2/\lambda$. Then the level of riskiness that an individual accepts for a random gain, which turns out to be neutral with respect to the exponential utility function (10), is in inverse proportion to her/his constant local risk aversion index $(1/\lambda)$. In other words the following two behaviors are equivalent:

- Determining (C, m) in such a way that the corresponding gain is neutral, with respect to utility function with constant $(= 1/\lambda)$ local risk aversion index.

- Determining (C, m) so that the riskiness level is $\lambda/2$.

We take this opportunity to recall that the function $(-u''/2u')$, has been later considered, independently of de Finetti, by Kenneth J. Arrow and John W. Pratt as a measure of *local degree of risk aversion* and is named after them in the current literature. See [1] and [27]. As a matter of fact, this is not the sole circumstance in which de Finetti, who humbly wrote that he did not consider himself as an economist, was able to anticipate and develop ideas later rediscovered by renowned economists who had even been awarded the Nobel Prize such as K. J. Arrow in 1972. A recent letter by K. J. Arrow on this subject can be found on the web-page [33]. A similar situation occurred with Harry M. Markowitz, another Nobel Prize winner in 1990, who recognized in [25] de Finetti's priority in applying the mean–variance approach to finance. Apropos of this, see also [28] and references therein. It is worth noting that both these de Finetti's contributions to economics are intimately connected with his paper [13] where he makes a pioneering use of martingale-like probabilistic tools.

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