



“Mathématique Sociale” and Mathematics.

A case study: Condorcet’s effect and medians.

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Résumé

Condorcet a découvert l’ “effet Condorcet”, c’est-à-dire le fait qu’agrèger des préférences individuelles transitives au moyen de la règle de majorité sur les paires de candidate peut conduire à une préférence collective qui n’est pas transitive. La solution proposée par Condorcet pour pallier cet effet a été interprétée comme la recherche d’une médiane dans un certain espace métrique. Je parcours dans ce papier les nombreux domaines de mathématique "pure" ou "appliquée" où la notion de médiane (métrique) est apparue. S’il y avait réellement besoin d’une preuve que les mathématiques sociales sont bien des mathématiques, le cas de la médiane en fournirait une convaincante.

Abstract

L’ “effet Condorcet” refers to the fact that the application of the pair-wise majority rule to individual preference orderings can generate a collective preference containing cycles. Condorcet’s solution to deal with this disturbing fact has been recognized as the search for a median in a certain metric space. We describe the many areas of "applied" or "pure" mathematics² where the notion of (metric) median has appeared. If it were actually necessary to give examples proving that “social mathematics” is mathematics, the median case would provide a convincing example.

1. Introduction

The expression "Mathématique Sociale" has been coined by Condorcet in order to designate "*la science qui a pour objet l'application du calcul aux sciences politiques et morales*" (Tableau general, 1793) and distinguish this science from

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² I don’t believe that there are distinct "pure" and "applied" mathematics. There are mathematics motivated by internal questions to mathematics and mathematics motivated by questions raised in other sciences. But, as the present paper will illustrate it, these two kinds of motivations can lead to very intercrossed mathematics. Another striking example is in [Kuhn, 1976].

Petty's "Political Arithmetic" and Buffon's "Moral Arithmetic". What Condorcet meant by "Mathématique Sociale" has been studied by several authors (see, in particular, [Granger, 1956], [Baker, 1975] and [Crepel and Rieucan, 2005]). Here I will use this term for the mathematics used in some models or methods of the human and social sciences, for instance the mathematics used in mathematical economics, mathematical psychology or mathematical sociology. Observe, moreover, that one can also include in "Mathématique Sociale" the fact that mathematics is a social object that can be studied from different points of view by historians, economists, psychologists or sociologists.³ The point of view of these social scientists could be useful for explaining the assertion sometimes heard that "La Mathématique Sociale, c'est pas des maths" (Social Mathematics is not Mathematics) or some weakened assertions like "it is not interesting mathematics". Such peremptory and obviously false assertions do not really deserve to be contradicted. Nevertheless, I will develop in this paper what can be considered as a counter-example to this kind of judgment. Indeed, I intend to show how Condorcet's method (according to Guilbaud and Young) to deal with "l'effet Condorcet" uses the notion of median, a notion developed in many domains of "pure" and "applied" mathematics. Moreover, Condorcet's median procedure has raised and is still raising interesting mathematical problems.

In section 2, I recall the method used by Condorcet to aggregate preferences, what "l'effet Condorcet" is and why the Condorcet solution to deal with this effect leads to a difficult combinatorial optimization problem. In section 3, I define the notion of median in a metric space and I explain why the Condorcet solution is a (metric) median. Section 4 sketches a history of the metric median from Fermat to Birkhoff and later, history in which we meet on the way Jordan, Lebesgue, Weber, Gini and many other mathematicians. Section 5 skims over the theory of the "good" discrete metric spaces for medians. In the final section, I come back to Condorcet by showing that several ways found to avoid the Condorcet effect rely on results obtained in section 5.

2. Condorcet's solution to deal with "l'effet Condorcet"

First, I recall Condorcet's method for aggregating (judgments of) preferences.⁴ I

³ This meaning of "Mathématique Sociale" was well present in the perspectives of Guilbaud when he created the Centre de Mathématique Sociale (see Monjardet 2005 for the creation of this Center and its activities in social choice theory).

⁴ It is worthwhile to recall that Condorcet's aim is much more general. He want to aggregate *coherent opinions*: let a set of binary ("yes or no") questions be logically linked in the sense that the answers to some imply the answers to others; a *coherent opinion* of an individual is a set of answers to these questions respecting their links. In particular a coherent opinion can be the answers to questions bearing on the culpability of an accused. And one finds in the *Essai* an

will use the classical notations of social choice theory. The mathematical model consists of a set of "voters" each one having a preference on a set of "candidates". This preference is expressed by a *ranking*⁵ and the specification of all these rankings forms the *profile of preferences* of the voters. For instance, a profile of preferences of 7 voters on 4 candidates denoted by a , b , c and d is the following:

1	2	3	4	5	6	7
c	c	c	b	b	a	a
b	b	b	a	a	d	d
a	a	a	d	d	c	c
d	d	d	c	c	b	b

A profile of preferences

In the above profile, voter 1 has the preference $c > b > a > d$ and voter 4 the preference $b > a > d > c$. A preference ranking like $c > b > a > d$ expresses six binary preferences between two candidates: $c > b$, $c > a$, $c > d$, $b > a$, $b > d$ and $a > d$.

In the *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix* (henceforth, called simply the *Essai*) Condorcet uses the majority rule on these binary preferences to get a collective preference.⁶ For the above profile the following table shows the number of votes obtained by each binary preference i.e., what can be called the *support* given by the voters to this preference:

$x > y$	a	b	c	d
a		2	4	7
b	5		2	5
c	3	5		3
d	0	2	4	

The binary preferences of the above profile

example of the so-called "doctrinal paradox" rediscovered two hundred years later (see the *Discours préliminaire* of the *Essai*, pages 50-53, and see [Guilbaud, 1951] and [Granger, 1965] for an analysis of Condorcet's work on the logical problem of aggregation.

⁵ Here a ranking means a *strict linear order* i.e., an *asymmetric*, *connected* and *transitive* binary relation.

⁶ This method had been already proposed in the thirteenth century by Ramon Llull (see [McLean and London, 1990] and [McLean, Lorrey and Colomer, 2008])

One reads in this table that, for instance, a is preferred to c by 4 voters and that c is preferred to d by 3 voters. The bold figures are those which are greater or equal to the majority 4.

Condorcet retains these bold figures to get the collective preference $a > c$, $a > d$, $b > a$, $b > d$, $c > b$ and $d > c$. But this preference does not form a ranking since it contains a cycle $b > a > d > c > b$. Equivalently, this preference is not transitive.⁷ This fact discovered by Condorcet has been called "l'effet Condorcet" par Guilbaud (1952) and the "paradox of voting" by Arrow⁸ (1951).

In the *Essai* Condorcet proposes a procedure for getting a ranking that expresses the collective preference when the majority rule induces a nontransitive preference. He writes: "on écartera de l'avis impossible successivement les propositions qui ont une moindre pluralité et l'on adoptera l'avis résultant de celles qui restent"⁹ ("one successively deletes from the impossible opinion the propositions that have the least plurality, and one adopts the opinion from those that remain").

However Condorcet's sentence is ambiguous (at least when there are more than three candidates) and can allow for different interpretations (see [Black, 1958]). Here I adopt the interpretation shared by Guilbaud (1952) and Young (1988).

Guilbaud writes: "Condorcet ne peut se résigner à conclure qu'on ne peut attribuer aucune opinion cohérente (*ordre total*) au corps électoral...Il cherche un moindre mal, c'est-à-dire parmi toutes les opinions cohérentes celles qui est appuyée par le plus grand nombre de suffrages" ("Condorcet could not resign himself to conclude that it is impossible to attribute any coherent opinion (*ranking*) to the electoral body (.....). He looks for lesser evil, that is to say among all the coherent opinions the one which is supported by the largest possible number of votes").¹⁰

What is the support given by the voters to a ranking that could represent the collective preference? It is clearly the sum of the supports of the binary preferences contained in this ranking. For instance, for the above profile the support of $a > b > c > d$ is $2+4+7+2+5+3 = 23$, whereas the support of $c > b > a > d$ is $5+3+3+5+5+7 = 28$. One can check that this order $c > b > a > d$ has the strongest support among the

⁷ When a binary relation is connected and asymmetric it is transitive if and only if it does not contain cycles.

⁸ According to Urken the fact that Arrow used the term paradox could be explained by the fact that he followed Tarski's teaching in logic. See also [Arrow, 1991] and [Suppes, 2005].

⁹ In this sentence *avis* (translated below by *opinion*) means the collective preference obtained by the majority rule and *proposition* means binary preference.

¹⁰ Page 265 in the English translation of Guilbaud's 1952 paper.

24¹¹ possible rankings on 4 candidates. So, according to Guilbaud, it is the Condorcet solution to the problem raised by the Condorcet effect encountered in this profile.

Two observations are necessary. First if the profile does not exhibit a Condorcet effect the majority rule gives the ranking with the strongest support.¹² Second the Condorcet solution is not necessarily unique: there can exist several rankings having the strongest support.

I now arrive at Young's analysis of the procedure proposed by Condorcet. He recalls that for Condorcet there is a true order of merit between candidates. The aim of the voting procedure is to find this order from those given by the voters and containing errors. Then Condorcet introduces a probabilistic model for the search of this true order: the probability for each voter to prefer the best candidate in each binary comparison is a constant $p > 1/2$. And Condorcet searches the ranking that is the most probable combination of propositions in this model i.e., what is now called the *maximum likelihood estimation* of the ranking. Now (modulo independence hypotheses) a simple computation shows that this ranking is "the set of propositions that contain no cycles and is supported by the largest number of pairwise votes" i.e., exactly the same ranking as the one given by Guilbaud. So, later on, I will call Condorcet's solutions or Condorcet's rankings the rankings having the strongest support.¹³

How to find this (or these) ranking(s) which is (are) the Condorcet solution(s) to the problem raised by the Condorcet effect? The answer is obtained by solving a *combinatorial optimization problem*. What is such a problem? One has a finite set of elements (here the elements are all the possible rankings); a numerical value is assigned to each element (here it is the support of the ranking). And one searches the element(s) having the greatest value. Such a problem can seem trivial: it is sufficient to list all the elements while keeping at each step the one with the greatest value. But it is not at all trivial as soon as the size of the set of elements is large. In Condorcet's problem the number of possible rankings for m candidates is $m!$ a number that expands rapidly.¹⁴ Combinatorial optimization problems abound in mathematics for instance in combinatorics, operational research or data analysis. Many such problems are very difficult –in fact impossible– to solve as soon as their size is not small. Then the combinatorial optimization theory (a branch of the optimization theory) has developed many approaches and tools to tackle such

¹¹ The number of different rankings of m candidates is $m! = m(m-1)\dots 2.1$.

¹² At least when the number of voters is odd. When it is even the (strict) majority rule gives a partial order and all the rankings containing this partial order have the strongest support.

¹³ See also [Crépel, 1970] for an analysis of several other less known Condorcet's texts on the elections which "confirm plutôt" ("rather confirm") Guilbaud's and Young's interpretations.

¹⁴ For $m = 15$, $m! = 1307674368000$ and for $m \geq 25$ $m!$ is greater than 10^m .

"insolvable" problems.¹⁵ For instance, instead of searching for the exact solution of the problem, one searches "good" approximate solutions. The search for the Condorcet rankings is a particular case of a classical "insolvable" problem called the *linear ordering problem* (see [Reinelt, 1985]) and it is itself an "insolvable" problem.¹⁶

In the next section we will see that Condorcet's rankings are also medians in a certain metric space.

3. Metric spaces and medians

We all know two notions of medians. In geometry, a median of a triangle is a line that joins a vertex of the triangle to the middle of the opposite side. In statistics, the median of a statistical distribution is its "middle": half the values are above the median and half are below it. Observe that this last sentence gives only an intuitive idea of what this median is. It must be specified according to the nature of the distribution and it is not uncommon to find imprecise or erroneous definitions in many manuals of statistics.¹⁷

In fact in order to show that Condorcet rankings are medians we are going to use a more general notion of median namely the notion of metric median in a metric space. I begin by saying what is meant by a metric space, a notion introduced by Maurice Fréchet in his 1904 doctoral dissertation and that is become a fundamental notion in mathematics.¹⁸ A *metric space* is a set of elements, often called points, endowed with a *distance*: a non-negative numerical value $d(P,Q)$ is associated to any pair $\{P,Q\}$ of points of this space and d satisfies the following properties:

$$\begin{aligned}d(P,Q) &= 0 \text{ if and only if } P = Q \\d(P,Q) &\leq d(P,R) + d(R,Q)\end{aligned}$$

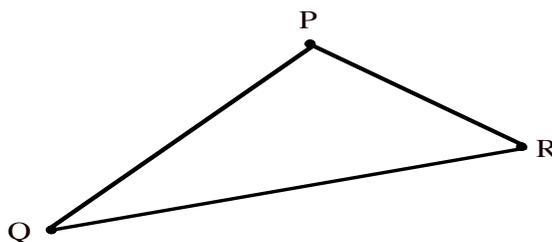


Figure 1

¹⁵ There exist many good books on combinatorial optimization, for instance *Combinatorial Optimization* by W. J. Cook, W.H. Cunningham, W. R. Pulleyblank and A. Schrijver, Wiley, 1997.

¹⁶ One can find a survey on the way to tackle the aforementioned problems in [Charon and Hudry, 2007].

¹⁷ For a correct handling see, for instance, [Hays, 1994].

¹⁸ The concept of metric space is due to Fréchet but the name is due to Hausdorff.

This notion of distance is obviously an abstraction of the usual distance in our Euclidean space. The second property called the *triangular inequality* reflects the fact that in this space the straight line is the shortest path from a point to another point (Figure 1). Metric spaces are particular topological spaces and their theory is a part of the foundation of modern mathematical analysis.

Let $P_1, P_2 \dots P_n$ be n points of a metric space. A point M is a *median* of these n points if M is a point of the space minimizing the sum $d(M, P_1) + d(M, P_2) + \dots + d(M, P_n)$ of its distances to the n points.

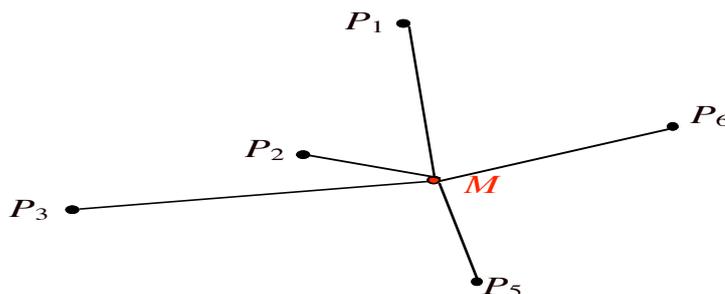


Figure 2 The median of six points

It is interesting to observe that the median of a statistical distribution can be defined like this. It is a result in [Laplace, 1774]: let F be the cumulative distribution function of an (absolute continuous) random variable; the median is defined as the value μ such that $F(\mu) = 0.5$. Laplace proved that μ is also the value minimizing the average of the absolute deviations, where the *deviation* between two values is their L_1 (called also *Manhattan*) distance i.e., the absolute value of their differences. Laplace called this value "le milieu de probabilité" or "la valeur probable". The term median has been introduced by Cournot in *l'Exposition de la théorie des chances* (1843, p.63 et 120)¹⁹.

In order to show that Condorcet's solutions are (metric) medians we have to introduce a metric space. We consider the set of all possible rankings on the candidates. For instance, if there are 3 candidates denoted by a, b, c there are 6 rankings shown below:

a	b	b	c	c	a
b	a	c	b	a	c
c	c	a	a	b	b

¹⁹ Many terms have been used to name a metric median: Fermat point, Fermat-Torricelli point, Fermat-Weber point, Steiner point, Lamé point, equiprobable value, minimum distance point, minimum travel point, proximal point, center, centroid.... The use of some of these terms is linked to particular contexts where metric medians have appeared (see sections 4 and 5).

In this space of all the rankings we introduce a very natural distance called the disagreements' distance. There is a *disagreement* between two rankings on a pair $\{x,y\}$ (of candidates) if they express opposite preferences on these candidates: in one ranking x is preferred to y and in the other y is preferred to x . The disagreements' distance between two rankings is the sum of their disagreements on the pairs. For instance, the distance between the two rankings $a>b>c$ and $b>c>a$ is 2 since they disagree on $\{a,b\}$ and $\{a,c\}$.

It is easy to check that the disagreements' distance is indeed a distance in the above sense i.e., that it satisfies the triangular inequality.²⁰ One can represent the metric space of all the rankings by a graph called the *permutoèdre graph* where the distance between two rankings equals the distance of the shortest path between the two vertices representing these rankings. Figures 3 and 4 below show the permutoèdre graph for 3 and 4 candidates. In these figures a ranking is represented by a *permutation*. For instance, $abcd$ represents the ranking $a>b>c>d$. One sees on Figure 4 that the distance between, for instance, $abcd$ and $bdca$ is 4.

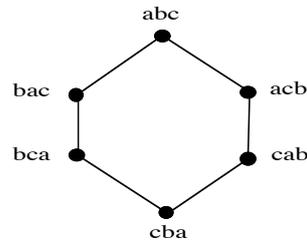
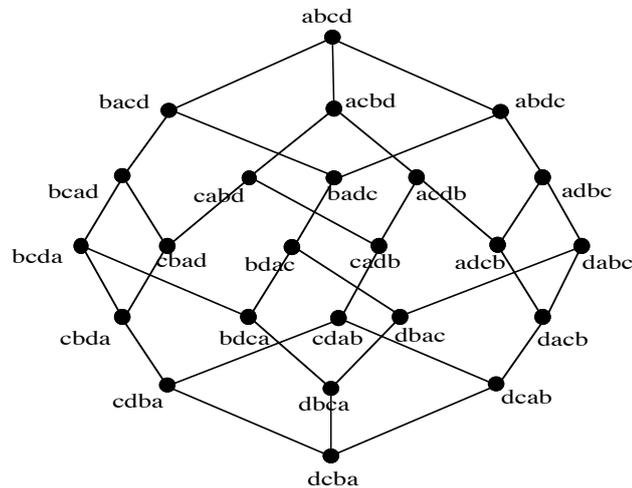


Figure 3: the permutoèdre graph for 3 candidates.



²⁰ This distance is nothing else than the half of the well-known distance between binary relations (more generally between sets) called *the symmetric difference distance* i.e., the number of elements that belong to one of these two relations (sets) without belonging to the other. For two binary relations R,S their symmetric difference distance is thus $|R-S| + |S-R|$.

Figure 4: the permutoèdre graph for 4 candidates.

Since we have endowed the set of all rankings with a metric space structure, we can speak of medians in this space. A ranking M is a median ranking of n rankings $L_1, L_2 \dots L_n$ if M minimizes among all the possible rankings the sum $d(M, L_1) + d(M, L_2) + \dots + d(M, L_n)$ of its distances to the n rankings.

Defining a median ranking as the consensus between n rankings is the aggregation procedure proposed by Kemeny (1959) and reproduced in Kemeny and Snell's 1962 book on the mathematical models of social sciences. This procedure is often called the Kemeny rule. Now we have the following result ([Barbut, 1966]): let $(L_1, L_2 \dots L_n)$ be a profile of n rankings. A ranking M is a median ranking of this profile if and only if it is a Condorcet's solution (i.e., if M has a maximum support among all the possible rankings). In other words Kemeny's solution to the aggregation problem is the same as Condorcet's solution.

This result shows the equivalence of two formulations for defining a consensus ranking of a profile of rankings. In fact, one can find many other equivalent formulations for this same ranking (in Monjardet (1990) one will find twenty-two equivalent formulations). In particular, it is interesting to observe that this same median ranking was independently proposed in 1960 under two other different formulations and names: by Brunk who adapted an idea in Kendall (1942) and by Hays using Kendall's tau to define a consensus ranking. It is beyond doubt that it is the big success of Kemeny and Snell's book that made Kemeny's rule so popular, whereas Brunk's and Hays' works remained largely ignored.

In the next section I will show that the notion of (metric) median has appeared in many domains of mathematics.

4. Medians in mathematics

Apparently, the history starts with Pierre de Fermat. At the end of his *Essai sur les maximas et les minimas* (1629, p153 in [Oeuvres]) Fermat launches the challenge: "Let he who does not approve of my method attempt the solution of the following problem: given three points of the plane, find a fourth point such that the sum of its distances to the three given points is a minimum". So the question is to find the median of three points of the plane for the usual "straight line" (Euclidean) distance. The answer²¹ is not in Fermat's *Essai* but from his formulation of the problem one can assume that he knew it. One finds the answer first in Torricelli's

²¹ This answer is not the intersection point of the three medians of the triangle. This intersection point is the *center of gravity* of the triangle, i.e. the point minimizing the sum of the square of the distances to the three vertices of the triangle. This confusion between median and center of gravity is frequent (a famous example is given at the end of this section).

and Cavalieri's works (1647 and 1648). When no angle of the triangle is greater or equal to 120° , Torricelli proves that the median is the intersection point of the three circles circumscribing the equilateral triangles constructed on the sides of and outside the given triangle. It is why this median is also called the *Torricelli point*. Cavalieri shows that this median point is the point from which the three lines joining it to the three vertices of the triangle form three angles of 120° . He also considers the case where the given triangle has an angle greater or equal to 120° . In this case the median is the vertex of this angle (a result obtained again by Heinen in 1834). Later Simpson (*Doctrine et applications des fluxions*, 1750) observes that the median point is also the intersection point of the *Simpson lines* i.e., the lines joining the outside vertices of the equilateral triangles considered by Torricelli to the opposite vertices of the given triangle.

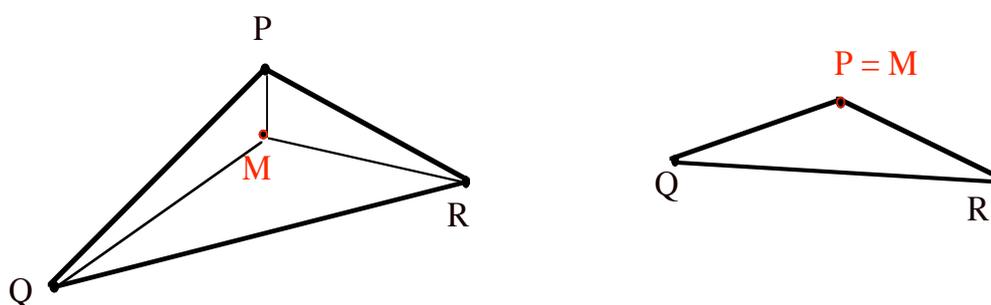


Figure 5: the median of three points of the plane.

Generalizations of Fermat's problem occur as early as an exercise in Simpson's book. Indeed, Simpson adds numerical weights to the points and searches the median of 3 weighted points i.e., the point M minimizing the sum $pd(M,P)+qd(M,Q)+rd(M,R)$. The solution would appear in [Launhardt, 1872]. It is interesting to observe that Lebesgue (1918) devoted a lesson to this problem where he insists that the existence of the median must be proved. The problem of searching for the median of n points in the plane appears for sure in Steiner (1838), which has perhaps also raised the problem for n weighted points. These problems are discussed by Sturm (1884), which shows the unicity of the median when the points are not aligned. But as soon as the number of points is greater than 4 it becomes very difficult to find the median point.²² I will not try to quote all the many works that may have appeared on the (generalized) median problem in the plane or in

²² In fact one can only find "good" approximation algorithms. See [Weiszfeld, 1934], [Ostresh, 1978]), [Eckhardt, 1980], [Bajaj, 1988] or [Chandrasekaran and Tamir, 1990] for algorithms and complexity results on this problem.

various spaces before or during the 20th century.²³ But most of the works that appeared before the fifties remain largely ignored or were despised. For instance in 1941 two illustrious mathematicians Courant and Robbins dealt with Fermat's problem (called by them Steiner's problem...) in their famous book *What is mathematics?* But concerning the problem of the median of n points they wrote: "This problem which was also treated by Steiner does not lead to interesting results. It is one of the superficial generalizations not infrequently found in the mathematics literature". Unfortunately, this peremptory judgment was already completely wrong.²⁴ Indeed, the interest of this problem has appeared at the beginning of the century in two different contexts (admittedly not in pure mathematics).

In 1909 Alfred Weber (economist and sociologist, brother of Max Weber) published *Über den Standort der Industrien (On the location of industries)*. This book was the first to present a mathematical model for the optimal location of a plant. In order to build a product the plant receives two types of raw material located in two different places. The product must be sold at a third place. Weber's model takes in account the distances between the different places and the costs of the transportations. The plant's location must minimize the costs, but since Weber assumes that it can be anywhere in the space, one finds again Simpson's problem of the research of the median of 3 weighted points. Since Weber and particularly since the sixties many more general models of "continuous" location have been studied. Then the works of Fermat, Toricelli and others have been rediscovered, and so, in facility location theory, one now currently speaks of the Fermat-Weber generalized problem (see [Kuhn, 1967,1976]).

Another generalization of Fermat's problem also took place in statistics at the beginning of 20th century. The story starts in 1914 but it became well known only in 1930. This year, the famous *Journal of the American Statistical Association* published "A mistaken conception of the center of population" a paper by the

²³ One will find many references going from Fermat to recent works on Fermat's problem and its generalizations in the Fermat-Toricelli entry of the Springer Encyclopedia of Mathematics: <http://eom.springer.de/f/f130050.htm>. See, in particular, [Kupitz and Martini, 1997] and [Wesolowsky, 1993]. There exists also a dual problem of Fermat's problem of which the origin seems to come back to a 1811 paper by Rochat, Vecten, Fauguier and Pilatte (see [Kuhn, 1967 and 1976]).

²⁴ One can observe that in the 1962 German edition of the book the judgment is a little less peremptory: "Diese Problem, das ebenfalls von Steiner behandelt wurde, führt nicht zu besonders interessanten Ergebnissen. Es ist eine der oberflächlichen Verallgemeinerungen, wie man sie in der mathematischen Literatur manchmal antrifft". Also, Courant and Robbins made wrong remarks about what they called a "complementary problem" of Steiner's (in fact Fermat's) problem (see, for instance, [Krupur, 1998], [Martini and Weissbach, 1999], [Jalal and Krupur, 2003]).

statistician Eells. He pointed out that since 1910 the United States Census Bureau committed “an unfortunate error” by trying to explain how is defined the center of the American population. This Bureau confused the notion of center of gravity (see footnote 21) and the notion of median of a population. Then, Eells points out that it is much more difficult to find the median than the center of gravity. He continues by solving the problem of finding the median of an equilateral triangle, i.e. a very particular case of the case solved by Fermat, Toricelli and Cavalieri almost two hundred years before!! This Eells’ paper aroused an avalanche of letters and in the next issue of JASA the chief-editor was obliged to do a synthesis of the received letters. The most interesting was the letter of the Italian statistician Corrado Gini that mentioned a paper by himself and Galvani published in 1929 in *Metron*. In this paper Census Bureau’s error was already mentioned. But, above all, the paper contained the proof of the existence of a solution for the problem that the authors called the Fermat-Toricelli’s generalized problem: to find the median of n weighted points of the Euclidean plane. Moreover, this work have been preceded and motivated by Gini’s 1914 paper: "L’uomo medio". One knows that Quetelet had proposed to define an "homme moyen" of a population of men by taking the means of different characteristics (size, weights, quantified aptitudes...) describing this population. But it was quickly objected, in particular by Cournot (see [*Exposition de la théorie des chances et des probabilités*, 1843], p.213) and Poisson, that such a mean man could be an impossible man. Gini considered a generalization of Quetelet’s problem: how to define the central value of a multivariate statistical series? Such a series can be represented by a (weighted) cloud of points in the Euclidean space. And Gini’s answer was to define the central value as the (metric) median in this metric space. It is interesting to observe that the same answer was given later by Fréchet in his 1949 paper "Réhabilitation de la notion statistique de l’homme moyen" ("Rehabilitation of the statistical notion of the mean man"). Fréchet tackles the same problem as the problem studied by Gini, namely to define central values for multidimensional statistical series. In fact, Fréchet made many contributions in order to define what the *typical* values of abstract random elements in abstract spaces are (see, for instance [Fréchet, 1948]). Then he was able to give several solutions to the problem, the main one being the same as Gini’s solution i.e., the median(s) in a metric space. ²⁵

In the Fermat-Weber location problem as well as in the Gini-Fréchet central value problem the median is defined in a continuous metric space, for instance in

²⁵ Fréchet does not quote Gini’s works in his paper on "l’homme moyen". He only writes that it is after having heard a talk recalling the failure of Quetelet’s "homme moyen" that he had the idea to apply some of his researches to Quetelet’s problem. One cannot be sure but since Gini was a reputed statistician Fréchet could have known his works.

an Euclidean space. Condorcet's median is defined in a discrete space (since the number of all the possible rankings forming this space is finite). In the next section we will see that the occurrence of medians in several discrete problems and structures has led to a theory of the "good" discrete metric spaces for medians.

5. Medians in discrete metric spaces and median algebras

In 1869 Camille Jordan publishes a paper "Sur les assemblages de ligne" motivated by his researches on the structure and the symmetries of quadratic forms. Jordan associates a (finite) *undirected graph* to such a form. An undirected graph – called *assemblage* by Jordan – is formed by vertices joined by (possibly multiple) edges. A natural distance between two vertices in the graph is *the shortest path distance* i.e., the minimum number of edges of a path linking these two vertices.²⁶ Then endowed with this distance, a graph becomes a (discrete) metric space and one can speak of the median(s) of a set of vertices. Jordan was interested in the search of what he called the *centers* of the graph associated to a quadratic form. He calls *assemblage à continuité simple* what is now called a *tree* i.e., a graph connected and without cycles²⁷ (see an example at Figure 6 below). In this case Jordan shows that there are either one center or two centers linked by an edge and that they can be easily found. But one can show that the Jordan centers of a tree are exactly the (metric) medians of all the vertices of this tree²⁸. More generally one can search the medians of a set of vertices of a tree. Here also it is easy to find the medians and one shows that they always form a *path* in the tree. For instance in the tree of Figure 6, the medians of the set $\{a,b,c,i,l,n\}$ of vertices are the vertices d , e and h (here the distance between, for instance, d and l is 4). . So a tree is a first example of a discrete metric space where it is easy to compute medians. We will see later that this is due to the fact that trees can be defined by means of a ternary algebraic operation satisfying specific properties.²⁹

²⁶ A *path* is a sequence of vertices linked by edges. It is clear that the shortest path distance satisfies the triangular inequality. The other property for a distance is satisfied if (and only if) the graph is *connected* i.e., if there exists always a path between any two vertices of the graph.

²⁷ When the first and the last vertices of a path coincide, it is called a *cycle*.

²⁸ In fact, Jordan defines four kinds of centers. A (Jordan) *center of second kind* is what is (generally) now called a *center* in graph theory (i.e., a vertex minimizing the maximum distance between a vertex and the other vertices). A (Jordan) center is (as just said) a median of all the vertices of the tree, but it has also called an *absolute 1-median* and a *mass center*.

²⁹ This operation is precisely this associating to three elements their median. Its properties were first studied by [Avann, 1948] and [Scholander, 1952, 1954] and independently –under the name of *tree algebra*– by [Nebesky, 1969]. Nebesky's monograph is a mathematical work motivated by the use the so-called *projective tree in dependency conception* in syntax.

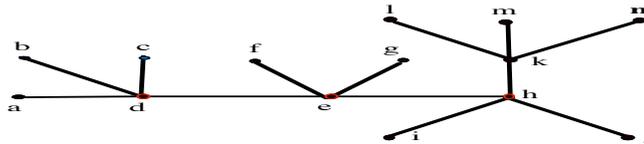


Figure 6: medians in a tree

If one now considers the problem of searching the medians of a set of vertices of an arbitrary (undirected) graph, it becomes less simple.³⁰ The problem has been very frequently studied since the sixties in the framework of the theory of discrete facility location models. There one searches the medians (called also the *Weber points*) of weighted vertices of a graph representing a location network model. So we are in a domain of applied mathematics in fact the domain of operational research.

But medians have also appeared in a domain of pure mathematics namely *lattice theory*.³¹ In 1947 Birkhoff and Kiss published a paper "A ternary operation in distributive lattices".³² A distributive lattice is a metric space with respect to a natural distance defined between its elements.³³ Then one can speak of the medians of a set of elements of this lattice. In their paper Birkhoff and Kiss show that the median of 3 elements of a distributive lattice can be defined by an algebraic formula using the operations meet and join of the lattice. So they obtain a ternary operation satisfying some properties and they show that such an operation characterizes distributive lattices. Let me explain what is this ternary operation in the simplest case of the distributive lattice defined by a ranking (see footnote 32).

³⁰ This is illustrated by a Slater's result: one can find graphs for which the set of medians of all their vertices is a given arbitrary graph [Slater, 1980].

³¹ A lattice is a partially ordered set –henceforth called a poset- in which any two elements have a *meet* (greatest lower bound) and a *join* (lowest upper bound). Lattice theory began at the end of 19th century with Dedekind and Schröder but it was at first largely ignored and it only rebirthed in the thirties with Birkhoff, Öre and many other mathematicians.

³² Lattices can also be defined algebraically by the properties of the two operations meet and join. When each one of these operations is *distributive* relatively to the other the lattice is said distributive. A simple example of distributive lattice is the set of all subsets of a set ordered by set inclusion. The natural distance between two elements of this lattice i.e., between two subsets is their symmetric difference distance (see footnote 20). Here the meet (respectively, join) of two subsets is their intersection (respectively, their union). In the lattice of subsets of a set there is also the operation of complementation of a subset that makes of this lattice a *Boolean lattice*. A still simpler example of distributive lattice is the linear order defining a ranking. Here the meet (respectively, the join) of two elements is their minimum (respectively, their maximum).

³³ This distance is the shortest path distance is the (undirected) *neighborhood graph* of this lattice: an element is a neighbor of another if it just below or just before it in the order of the lattice. For instance in the lattice of subsets of a set, a subset is neighbor of another if it is obtained from this other by deleting or adding a single element.

Take for instance the set of integers $\{1,2,\dots,9\}$ ranked by the usual linear order $<$. If one considers the series $4<5<8$ its median is 5. Now one can check that

$$5 = \text{Max}[\text{Min}(4,5), \text{Min}(5,8), \text{Min}(4,8)] = \text{Min}[\text{Max}(4,5), \text{Max}(5,8), \text{Max}(4,8)].$$

More generally, for $x<y<z$ the median is y and one has

$$y = \text{Max}[\text{Min}(x,y), \text{Min}(y,z), \text{Min}(z,x)] = \text{Min}[\text{Max}(x,y), \text{Max}(y,z), \text{Max}(z,x)].$$

This same formula of computation of the median of 3 elements is true in any distributive lattice when one replaces the operations Min and Max by the more general operations Meet and Join. On the other hand these algebraic formulas giving the median of three elements can be generalized to find the median of n elements.³⁴

Similarly using the ternary operation defining a tree (see footnote 29) one can find algebraic formulas for obtaining the medians of a set of vertices of this tree. This fact makes easy the computation of medians in trees like in distributive lattices. This observation led ("pure") mathematicians to search the more general discrete structure where computing the medians remains easy. This structure (that contains the trees and the distributive lattices) can be defined equivalently as a *median semilattice*, a *median algebra* or a *median graph*.³⁵ Other motivations came from ("applied") mathematicians working in consensus theory or (and) in graph theory have led to study other aspects of these structures, for instance axiomatic characterizations of the median (consensus) operation.³⁶

In the next section I return to the Condorcet effect. Indeed, I show how one can avoid this effect by using the fact that distributive lattices are good metric spaces for the medians.

6. How to avoid the Condorcet effect?

Since the set of all the subsets of a set is a distributive lattice (see footnote 32) the set of all binary relations defined on a set is a distributive lattice.³⁷ So, it is easy

³⁴ When n is odd there is a unique median. When n is even there can exist several medians but they form a *median interval* whose the two bounds are given by algebraic formulas (see [Barbut, 1961] and [Monjardet, 1980]). We will often speak of the median even when we will point out in a footnote the cases where it can exist several medians.

³⁵ See, for instance, [Avann, 1948], [Scholander, 1954], [Bandelt and Hedlikova, 1983], [Barthélemy and Bandelt, 1984].

³⁶ See for instance [Leclerc, 1990], [McMorris, Mulder and Roberts, 1998], [Hudry, Leclerc, Monjardet and Barthélemy, 2006] and the monograph [Day and McMorris, 2003] that contains many other references.

³⁷ A binary relation on a set X is nothing else that a subset of the set X^2 of all ordered pairs of X . Then the natural order between relations is the inclusion order: $R \subseteq R'$, i.e. xRy implies $xR'y$. The meet (respectively, join) of two relations is their intersection (respectively, union).

to find the medians of n arbitrary relations. In fact it is also easy to show that the median M of n relations $R_1, R_2 \dots R_n$ (with n odd) is given by the majority rule applied to these relations: xMy if and only if $xR_i y$ for at least $(n+1)/2$ relations.³⁸ Now if one take a set of binary relations satisfying some properties it is not generally a distributive lattice. It is obviously the case for the set of all rankings since two rankings are never comparable for the inclusion relation.

But it has been shown by Guilbaud and Rosenstiehl (1963) that the set of all rankings of m elements is a lattice for a certain order.³⁹ This lattice called the *permuttoèdre lattice* S_m has many interesting properties but it is not distributive.⁴⁰ Nevertheless one can find in S_m distributive sublattices. And even distributive sublattices which are "the same"⁴¹ as distributive lattices of partial orders, sublattices of the set of all binary relations. When one takes rankings in such a distributive lattice their median is a ranking obtained –like in the lattice of all the binary relations– by the majority rule.⁴² Then one obtains *restricted domains* of rankings where the majority rule works: for any profile of rankings taken in these domains there is no Condorcet effect. The search of such restricted domains has a long story beginning with the *single-peaked rankings* defined by Black (1958,1998).⁴³

As early as in 1952 Guilbaud observed that the set of all single-peaked rankings has a distributive lattice structure. Figure 7 reproduces a Figure in Guilbaud's paper showing the distributive lattice of all the single-peaked rankings of 5 candidates.

$$\begin{array}{c} \text{BACDE} \rightarrow \text{ABCDE} \\ \uparrow \\ \text{CBADE} \rightarrow \text{BCADE} \end{array}$$

³⁸ If n is even one can have several median relations forming an interval median whose bounds are given by applying the *strict* and the *large majority rules* (see footnote 34 and [Barthélemy and Monjardet,1981]).

³⁹ One fixes a reference ranking as the least ranking of this order. Then a ranking is less than another ranking if the set of its disagreements with the reference order is contained in the set of the disagreements of the second ranking (with the reference order). For instance, take 4 elements a,b,c,d and the reference ranking $dcba$ ($d > c > b > a$). Then one obtains a lattice that can be represented by Figure 4. In this Figure an edge represents now that a ranking is just below or just above another ranking in the order between rankings. For instance the edge from $bdac$ to $badc$ means that $bdac$ is just below $badc$ (since it has one less disagreement with the reference order $dcba$).

⁴⁰ See, for instance, [Caspard,2000].

⁴¹ i.e., that are *isomorphic* to

⁴² Here also, one must distinguish the case where n is odd and where the median is unique and the case where n is even that allow to have several medians (see footnote 34)

⁴³ Let $>$ be a "reference" ranking. A ranking L is single-peaked w.r.t. $>$ if for every ordered triple $x > y > z$ the middle element y is never ranked last in the restriction of L to $\{x, y, z\}$ (the term single-peaked comes from another characterization of these orders; see [Black,1958]). The rankings on Figure 7 are single-peaked w.r.t. the ranking $A > B > C > D > E$ ($= ABCDE$).

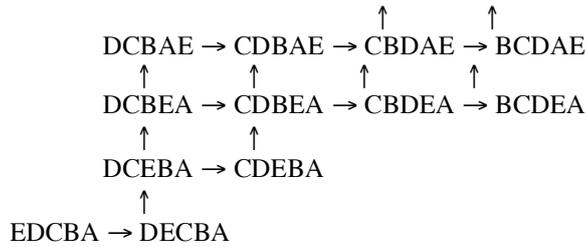


Figure 7: the distributive lattice of single-peaked rankings (for $m = 5$).

When there are m candidates there are 2^{m-1} single-peaked rankings of these candidates. Other restricted domains not exceeding this size were found and for a while it was conjectured that it was the maximum size of a restricted domain. But this was a completely wrong conjecture since it is false for $m \geq 4$. In particular, there exists a restricted domain of size 9 ($> 2^3 = 8$) for 4 candidates. Since Black, many works have been dedicated to this topic. In particular it has been shown recently that most of the discovered restricted domains were distributive lattices of the kind described above.⁴⁴ Just below in Figure 8, I give another example of such a distributive lattice of rankings. It is a restricted domain of size 45 that is the maximum size of a restricted domain for 6 candidates.⁴⁵

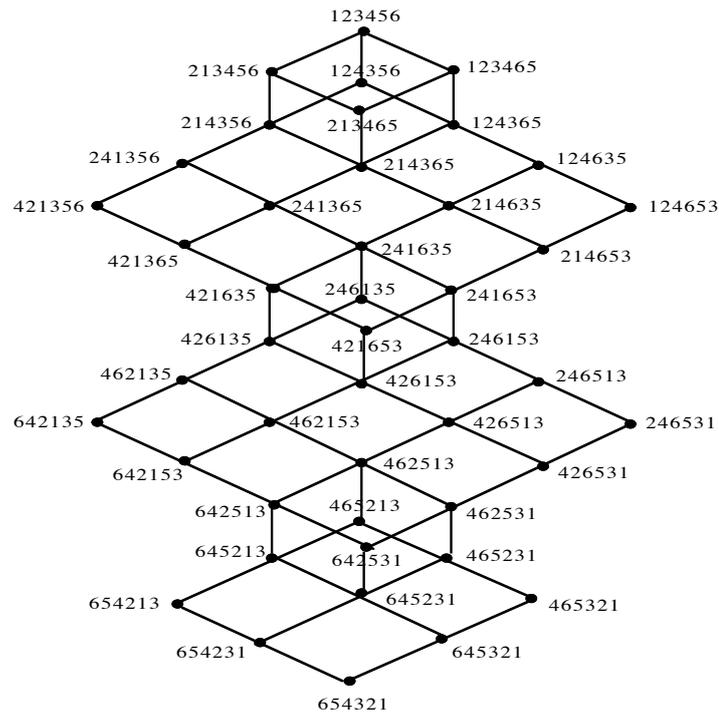


Figure 8: a restricted domain of maximum size sub-distributive lattice of the permuttoèdre S_6

⁴⁴ One will find a story of the works on the restricted domains with the last results due to [Galambos and Reiner, 2008] in [Monjardet, 2008].

⁴⁵ I found this example which was published in [Chameni-Nembua, 1989] but it was only in 1997 that Fishburn proved that it was a restricted domain of maximum size (for $m = 6$).

7. Synthesis and conclusion

Condorcet's problem considered in several sections of this paper is a problem of aggregation of individual preferences into a collective preference. Many similar aggregation problems have been studied in data analysis.⁴⁶ For instance one can have several classifications of a set of objects and one searches a consensus classification. Here too, one can often define a distance between two classifications and then search a median in the induced metric space on the set of all classifications. This approach has for instance been proposed by Régnier (1965) and Mirkin (1979) when the classifications are partitions⁴⁷ of the set of objects. But computing a median partition is as difficult a problem as computing a median ranking (except of course if one works in restricted domains of partitions that are distributive lattices⁴⁸).

In this paper we have seen that the notion of median is used in many areas of "pure" and "applied" mathematics. In order to gain a better understanding of the similarities and differences between the many uses of this notion, I will be a little bit more abstract. I consider complex elements formed by elementary elements, for instance objects described by several attributes. I denote by $x = (x_1, x_2 \dots x_p)$ such an element and I will refer to the elementary elements x_i as the *coordinates* of the element x . Each coordinate x_i takes his value in a set A_i which, keeping on the Euclidean space analogy, can be called a *coordinate axis*. In many cases the sets A_i will be the set \mathbf{R} of real numbers or a subset of \mathbf{R} , for instance the set of integers or simply the set $\{0,1\}$. I consider the set of all possible elements $x = (x_1, x_2 \dots x_p)$ when one assumes that each coordinate can take -independently of the other coordinates- any value in A_i . This set is called the *direct product* of the sets A_i and it is denoted by $D = A_1 \times A_2 \times \dots \times A_p$ or, if $A_1 = A_2 = \dots = A_p = A$ by A^p . Now as soon that the sets A_i are ordered by orders denoted by \leq_i , the set A can be endowed with the *product order* $\leq : x = (x_1, x_2 \dots x_p) \leq y = (y_1, y_2 \dots y_p)$ if and only if $x_i \leq_i y_i$ for $i = 1, 2 \dots p$. And as soon as the posets (A_i, \leq_i) are distributive lattices the poset (D, \leq) is also a distributive lattice. In fact, I will consider only the simplest case where all the posets (A_i, \leq_i) are the same poset (A, \leq) where A is a subset of the set \mathbf{R} of real numbers endowed with the usual (linear) order between numbers. As already mentioned (see footnote 32) (A, \leq) is then a distributive lattice where the meet (respectively, the join) of two numbers is their minimum (respectively, their maximum). Therefore A^p is a distributive lattice with the two following

⁴⁶ See [Barthélemy and Monjardet, 1981 and 1988].

⁴⁷ A partition divides a set of objects into disjoint classes.

⁴⁸ The partitions of a set form a (not distributive) lattice for a natural order between them (see [Barbut and Monjardet, 1970]. See [Barthélemy and Leclerc, 1995] for the state of the art on the median consensus of partitions.

operations of meet and join:

$$\text{Meet}(x, y) = [\text{Min}(x_1, y_1), \text{Min}(x_2, y_2) \dots \text{Min}(x_p, y_p)]$$

$$\text{Join}(x, y) = [\text{Max}(x_1, y_1), \text{Max}(x_2, y_2) \dots \text{Max}(x_p, y_p)]$$

These two operations are illustrated Figure 9 when $A = \mathbf{R}$ and $p = 2$ i.e., for the distributive lattice \mathbf{R}^2 .

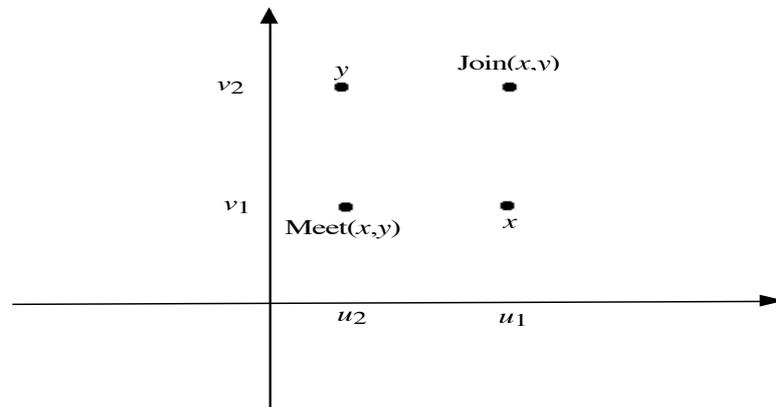


Figure 9 The meet (u_2, v_1) and the join (u_1, v_2) of $x = (u_1, v_1)$ and $y = (u_2, v_2)$ in \mathbf{R}^2

There are several ways to define distances in the distributive lattice A^p . One can use the usual Euclidean distance d_2 where for $x = (x_1, x_2, \dots, x_p)$ and $y = (y_1, y_2, \dots, y_p)$, $d_2(x, y)$ is the square root of $(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_p - y_p)^2$. But one can also choose as distance the L_1 (or *Manhattan*) distance d_1 defined by

$$d_1(x, y) = |x_1 - y_1| + |x_2 - y_2| + \dots + |x_p - y_p|.^{49}$$

In both cases A^p becomes a metric space and so one can speak of the median(s) of elements of this space. The generalized Fermat problem consists in searching for medians of sets of "points" in the Euclidean space, so with the distance d_2 . As mentioned above, when already in \mathbf{R}^2 this set contains more than 4 points the computation of a median is difficult. But one can also choose to search the median in the metric space defined by the L_1 distance. This was in fact one of Gini's proposals in the context seen above where the elements of \mathbf{R}^p represent statistical data. In this metric space the median is much easier to compute: indeed, the median of a "cloud" of points in this second metric space is obtained by taking the point whose the coordinate on each coordinate axis is the median of the coordinates of the cloud on this axis⁵⁰ For instance, in \mathbf{R}^3 the median for the L_1 distance of the 3 points $(3, 2, 7)$, $(1, 4, 8)$, $(2, 5, 6)$ is $(2, 4, 7)$.

Still with this L_1 metric space consider the particular case where the points of the cloud are $\{0-1\}$ -points i.e., points with only 0 and 1 as coordinates. What is the

⁴⁹ This distance is a particular case of the "natural" distance defined in a distributive lattice.

⁵⁰ So, when n is even it can exist several medians.

median on each coordinate axis? Since there are only two values the median value is the more frequent (majoritarian) value⁵¹. So, for instance, the median of the three points (1,0,0), (0,1,1) and (1,0,1) is (1,0,1). Now we can come back to the case where one must aggregate binary relations like rankings or equivalence relations.⁵² Why? Because any binary relation can be "coded" by a binary sequence of 0 and 1. Indeed, a binary relation on a set X of size s is a set of ordered pairs (x,y) . Let us rank all the s^2 ordered pairs of X in an arbitrary (linear) order and consider a binary relation R . It can be coded by a sequence $c(R)$ of s^2 numbers 0 or 1: we put 1 when (x,y) belongs to R (i.e., when x is in the relation R with y) and 0 if not. Then with this coding any binary relation can be seen as an element of A^p with $A = \{0<1\}$ and $p = s^2$. This direct product $\{0<1\}^{s^2}$ is a distributive lattice.⁵³ Let R and S be two binary relations on X and $c(R)$ and $c(S)$ the two sequences coding them in A^{s^2} . It is easy to check that the symmetric difference distance $|R-S| + |S-R|$ between R and S (see footnote 20) equals the L_1 distance between the sequences $c(R)$ and $c(S)$. What is the median of n binary relations in the space $\{0<1\}^{s^2}$? It results from what has been stated just above regarding the median of a cloud of $\{0,1\}$ -points that this median is the relation given by the majority rule on the ordered pairs.⁵⁴

In all the above cases we have implicitly assumed that given a set of elements of the distributive lattice A^p (with A subset of \mathbf{R}) its median(s) can be taken as representative element(s) of this set. The situation changes drastically if this hypothesis is wrong. It is the case when one considers only elements of A^p whose coordinates are not independent: there are numerical or logical links between these coordinates. Let S be a subset of A^p such that there exist links between the coordinates of its elements. When one takes the median of a set of elements of S , since this median is computed coordinate-wise, it may very well be that these links are no more respected by the median element. In other words the set S is not necessarily *stable* for the median operation. This is exactly what occurs with the Condorcet effect. In this case the set of all rankings on a set X is a subset of the distributive lattice formed by all the binary relations on X or equivalently, if one considers the above coding of binary relations, a subset of the direct product $\{0<1\}^{s^2}$. Now the coordinates of a ranking R in this direct product are not

⁵¹ So, there are 2^e medians where e is the number of coordinates where these two values are equal.

⁵² To aggregate equivalence relations is identical to aggregate partitions, since there is a one-to-one correspondence between partitions of a set and equivalence relations defined on this set.

⁵³ In fact this lattice is a Boolean lattice since it is isomorphic to the lattice of the subsets of the set X^2 of the ordered pairs of X (see footnotes 32 and 37).

⁵⁴ As already observed several times when n is even there can exist several medians (see footnote 34).

independent. Let c_{xy} , c_{yz} and c_{xz} be the coordinates of the ranking R for the three ordered pairs (x,y) , (y,z) and (x,z) . Then, it results from the transitivity of the ranking R that we must have $0 \leq c_{xy} + c_{yz} - c_{xz} \leq 1$. But to take the majority rule of a profile of rankings is the same as taking the medians independently on each coordinate axis of X^3 and then the links assuring the transitivity of the majority relation can be lost.

The problem of Quetelet's "homme moyen" is the same except that Quetelet used the arithmetical mean instead of the median. Since the mean of each characteristic of the population of men was computed independently from the other the mean man could be an impossible man. It is interesting to quote the already mentioned Cournot's objection: "Lorsqu'on applique la détermination des moyennes aux diverses parties d'un système compliqué, il faut bien prendre garde que ces valeurs moyennes peuvent ne pas se convenir : en sorte que l'état du système, dans lequel tous les éléments prendraient à la fois les valeurs moyennes déterminées séparément pour chacun d'eux, serait un état impossible"⁵⁵ ("When one applies mean operations to various parts of a complicated system, one must be aware that these mean values can be incompatible: the state of the system where all elements take the mean values separately determined for each could be an impossible state"). Cournot gave also the example of the triangle obtained by taking the means of the lengths of the three sides of rectangular triangles, a triangle that in general is not rectangular.

So, the Condorcet, Quetelet or similar aggregation problems bearing on complex elements whose coordinates are not independent (i.e. satisfy numerical or logical relations) arise because one computes representative elements (central values) of a n -tuple of these elements coordinates by coordinates. And the solutions found for these problems are the same. First one defines a distance between any two possible complex elements. Then, in the metric space of all possible complex elements, one computes the medians of the considered subset of elements. So, one makes sure that these medians taken as representative (complex) elements are possible (complex) elements. But now one may have a new problem: the median (complex) elements can be difficult to compute. It is in particular the case each time that the metric space of all the considered (complex) elements is not a "good" metric space for the median.

The considerations presented in this section have been already developed in the luminous Guilbaud's 1952 paper. First this paper reveals some previously ignored mathematics contained in Condorcet's *Essai*. Then it deals with the logical problem raised by the aggregation of complex objects when they are formed by elementary objects linked by logical relations. This problem is

⁵⁵ [Cournot, *Exposition de la théorie des chances*, p.213].

considered from Condorcet to Arrow by way of Laplace and the theory of errors, Quetelet, or Dom Quentin. So to complete this paper, the best thing I can do is to recommend Guilbaud's paper to the reader (some other comments on the same topic can be found in [Monjardet, 1991 and 2005]).

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