



*Journ@l Electronique d'Histoire des  
Probabilités et de la Statistique*

*Electronic Journ@l for History of  
Probability and Statistics*

Vol 9; Décembre/December 2013

**www.jehps.net**

## **On Two Historical Aspects of Buffon's Needle Problem**

PRAKASH GORROOCHURN\* AND BRUCE LEVIN†

### **Résumé**

Deux aspects du problème de l'aiguille de Buffon sont discutés. Premièrement, nous montrons que, contrairement à ce qui est habituellement défendu, Buffon n'a pas en fait utilisé l'expérience de l'aiguille pour estimer la valeur de  $\pi$ . Deuxièmement, et plus important, des précisions sont proposées sur le raisonnement suivi par Buffon pour déduire la probabilité d'intersection.

### **Abstract**

Two aspects of Buffon's needle problem are discussed. First, it is argued that, contrary to common lore, Buffon did not in fact use the needle experiment to estimate the value of  $\pi$ . Second, and more importantly, further clarification of Buffon's actual reasoning in deriving the probability of intersection is offered.

## **1 Introduction**

George-Louis Leclerc, Comte de Buffon (1707-1788) was born in Montbard, France, and was a major figure of the Enlightenment. Although he is known today foremost as a naturalist, Buffon showed interest in mathematics from an early age. He reportedly discovered the binomial theorem when he was 20 [Hanks, L., 1966, p. 17], unaware that Newton had already made the discovery in 1665. In that same year, he became friends with the illustrious Swiss mathematician Gabriel Cramer (1704-1752) who undoubtedly influenced his mathematical growth and development. At that time,

---

\* Department of Biostatistics, Columbia University, pg2113@columbia.edu

† Department of Biostatistics, Columbia University, Bruce.Levin@Columbia.edu

Buffon was a student while Cramer, only three years his elder, was already a professor. In his later *Essai d'Arithmétique Morale*, about which we shall have more to say later, Buffon said of Cramer:

...a man who has given us proof of his talent in all of the mathematical sciences, & to whose memory I do this justice, with even more pleasure since it is with the exchange and friendship of this Scholar that I have owed part of my first knowledge of this type... [Buffon, G.-L. L., 1777, p. 78]

Weil reports that both men were very passionate about mathematics [Weil, F., 1961]. In the letters that were exchanged between them, one thus finds enthusiastic discussions on indeterminate forms, algebra, geometry, and the St Petersburg Problem. Buffon had a vast number of other scientific interests which were evidenced in his monumental *Histoire Naturelle*, spanning 15 volumes between 1749 and 1767, and rivaling Diderot's *Encyclopédie*. However, it is his mathematical activities that concern us here.

Buffon lived in a period of intense mathematical activity. A century earlier, Newton and Leibniz had independently revolutionized the mathematical world through the discovery of the calculus. Newton had gone one step further: through the *Principia*, he had shown how the calculus could be used to understand our solar system and physical world. However, many continental mathematicians were still grappling with the idea of infinitesimal changes. Moreover, Newton's concept of gravitational force raised many questions. What was the nature of such a force and how could it obey a simple mathematical law? In his *An Essay Concerning Human Understanding* [Locke, J., 1690], Locke analyzed many of the existing doubts of the period in an attempt to understand the capacity of the human mind to comprehend the outside world. Both Berkeley's and Hume's answers were to express skepticism about the ability of mathematics to understand the physical world. Buffon had a keen interest in the type of questions Locke was asking, and his work in mathematics was really an attempt to clarify these. Thus, in the introductory essay of *Histoire Naturelle* [Buffon, G.-L. L., 1749], Buffon asks about the meaning of truth. For him there are different types of truth, the first of which are mathematical truths. However, Buffon says these are *abstract*

*truths* that are not connected to physical reality. In the *Essai d'Arithmétique Morale*, Buffon elaborates on his idea of probability and certainty. He distinguishes between physical and moral certainty. The former is based on experience, the latter on analogy. Buffon states that, although he has not seen the town of Constantinople, he is as certain of its existence as that of the moon, which he has seen many times. This is because of the experience of a huge number of witnesses and the corresponding certainty is a *physical certainty*. On the other hand, moral certainty is based on analogy which itself is the 'sum of the relationships of known facts'. Thus, if Buffon was told by a witness that a child had been just born in town, he would believe the witness completely. This is because the birth of a child has an 'infinite similarity' with the birth of all children. Now, if Buffon was told that this child had two heads, he would believe the witness again, but this time less strongly. Here the reasoning is by analogy to known facts and the corresponding certainty is a *moral certainty*. The latter is always much smaller than physical certainty. Buffon's concept of physical and moral certainty is thus seen to be similar to Hume's two types of truths, namely 'Relations of Ideas' and 'Matters of Fact.'

Buffon places moral certainty somewhere between doubt and physical certainty. He notes that there should be a number for the probability that represents moral certainty, but what is this number? Buffon proceeds as follows. From the mortality tables he finds that the probability of a man of 56 to die within the following 24 hours is about 1/10,000. However, an average health man of 56 is not afraid of dying the next day, so Buffon takes a probability of less than 1/10,000 as morally impossible and 'a probability of more than 10,000' as representing moral certainty.

In spite of his interest in mathematics, Buffon was more of a pragmatist than a mathematician *per se*. For example, his reaction to the St Petersburg problem\* is to reject the mathematician's solution of an expected infinite gain for the player, for two reasons. First he contends that, because all probabilities less than 1/10,000 should be regarded as zero, the player's "true" expected gain is much less than infinity. Second, he contends that, whereas the mathematician assesses the value of money by its

---

\* This problem was communicated to Buffon by Cramer.

quantity, the 'moral person' must gauge its value relative to what he already has. This reasoning is based on the concept of *expected utility* which is usually credited to Daniel Bernoulli [1738]. If Buffon's claim that his solution to the St Petersburg problem dated back to 1730 is correct, then he would have anticipated Bernoulli by eight years.

If Cramer had a big impact on Buffon's work in mathematics and probability, Newton had perhaps an even more profound influence on Buffon's scientific outlook. Newton's famous '*Hypotheses non fingo*'\* found a strong disciple in Buffon<sup>†</sup> and an advocate of the experimental method. Buffon was thus a proponent of the emerging empiricism that was to dominate the Enlightenment, and believed that any scientific theory should be the result of observable phenomena. Empiricism was characterized mainly through the sensation psychology of Locke and the scientific method of Newton. It thus represented a marked departure from the previous dominance of Descartes' rationalism where emphasis was placed on reason as a source of all true knowledge. Like many of his contemporaries of the Enlightenment, Buffon generally rejected metaphysical speculation in the natural sciences; in particular, he denied that God intervenes directly in nature. Equipped with a very powerful microscope, Buffon made careful studies of the abundant variety of life forms. He catalogued his observations in the *Histoire Naturelles*. He noted that, despite the variety and apparent disorder of life, nature is governed by fundamental laws. Buffon developed his own classification of living things. He recognized as valid groupings only species which were defined in terms of *reproductive capacity*. This was in marked contrast to the classification system of his Swedish contemporary Carolus Linnaeus, which was based on the idea of *resemblance*.

---

\* Newton's dictum was thus a departure from Descartes' prior system building. It must *not* be interpreted as a complete rejection of hypotheses though, since Newton himself framed several hypotheses in his scientific research. However, he was against the speculative formulation of reckless hypotheses without the support of experiment and observation.

<sup>†</sup> A reading of Buffon's preface in his French translation of the English version of Newton's "The Method of Fluxions and Infinite Series" clearly shows Buffon's thoughts on Leibniz's claim of priority on the Calculus: "Everybody knows that Leibnitz [sic] has wanted to share the glory of the invention, & many people still give him at least the title of the second Inventor..." [Buffon, G.-L. L., 1740, p. vi]

In the field of probability, Buffon is usually known for the *Needle Problem*. This was first written in 1733 in the *Mémoire sur le jeu du franc-carreau* while Buffon was trying to gain entrance in the *Académie Royale des Sciences*. The memoir features the Clean Tile Problem and the Needle Problem. In the Clean Tile Problem, a coin is dropped at random on a floor paved with tiles of various shapes and the probability of intersection with one, two, etc. tiles is sought. In the Needle Problem, a needle or “baguette” is thrown at random between two parallel lines and the probability that the needle crosses one of the lines is required. Although no solutions are provided, some hints are given regarding the Needle Problem and a statement is made that Buffon has solved the problem by using the area of a cycloid\*. The two problems together with an additional problem, namely the Grid Problem†, were later published with their solutions in 1777, in the *Essai d’Arithmétique Morale*. Gouraud reports that the *Essai* was actually written 17 years earlier [Gouraud, C., 1848, p. 54]. Interestingly, Buffon’s solution to only the first two problems is correct.

The Needle Problem is as follows: A needle of length  $2b$  is thrown at random on a plane in which a set of parallel lines separated by a distance  $2a$  (where  $a > b$ ) have been drawn (see Fig. 1). What is the probability that the needle will intersect one of the lines?

---

\* Defined as the curve traced by a point on the circumference of a circle which rolls on a straight line without sliding.

† The situation here is the same as in the Needle Problem, except that the needle is now thrown between two pairs of parallel lines.

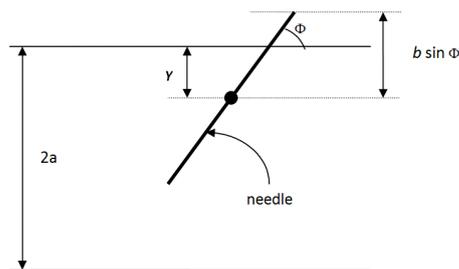


Figure 1 The Needle Problem

The problem is usually solved by either of two methods. As we shall soon see, neither of them corresponds to Buffon's own solution in the *Essai*. In the first method, let the distance between the midpoint of the needle to the nearest line be  $Y$  ( $0 < Y < a$ ), and the acute angle between the needle and the horizontal be  $\Phi$  ( $0 < \Phi < \pi/2$ ). We assume that  $Y \sim U(0, a)$  and that  $\Phi \sim U(0, \pi/2)$ , where  $U$  denotes the uniform distribution, and that  $Y$  and  $\Phi$  are statistically independent. Therefore,  $(Y, \Phi)$  is uniform on the rectangle  $\Omega = \{(y, \varphi) : 0 < y < a, 0 < \varphi < \pi/2\}$ . Now, the needle will intersect one of the lines if and only if  $Y < b \sin \Phi$  (see Fig. 1). The "favorable space" is then  $A = \{(y, \varphi) : 0 < y < b \sin \varphi, 0 < \varphi < \pi/2\}$ . Therefore, the probability of intersection is

$$\begin{aligned} \Pr\{Y < b \sin \Phi\} &= \frac{\text{area of } A}{\text{area of } \Omega} \\ &= \frac{\int_0^{\pi/2} b \sin \varphi \, d\varphi}{\pi a / 2} \\ &= \frac{2b}{\pi a}. \end{aligned}$$

Note that the above probability involves the number  $\pi$ , whose value can therefore be estimated by the experiment above. Thus, if the needle is thrown  $N$  times, of which a number  $n$  result in an intersection, then an appeal to the law of large numbers results in

$$\pi \approx \frac{2Nb}{na},$$

the approximation getting better, as  $N$  increases, with probability approaching one.\* In fact, the experiment was performed by several scholars such as Wolf, de Morgan, Lazzarani, and others [O'Beirne, T. H., 1965; Gridgeman, N. T., 1960; Gorroochurn, P., 2012]. But was it actually performed by Buffon himself? We return to this question below.

A second method of solution to the Buffon problem was later given by Barbier [1860].† Barbier finished his mathematical studies in 1860 and in that same year published his paper which also contained many other generalizations to the Buffon problem. He mistakenly credits the problem to Laplace's *Théorie Analytique des Probabilités*, and also acknowledges that the generalizations were taught by Mr Lamé in his lectures at the Faculty of Science. Barbier's solution is as follows. Suppose the needle is divided into two parts with arbitrary half-lengths  $b'$  and  $b''$ . Since a line intersects the needle if and only if it intersects either portion, the probability of intersection satisfies  $p(b) = p(b') + p(b'')$ . The latter equation is satisfied if and only if  $p(b) = kb$ , where  $k$  is a constant and  $b = b' + b''$ . Next imagine a polygonal line (not necessarily convex) of total length  $2b$  is made up of  $n$  rectilinear segments of half-lengths  $b_1, b_2, \dots, b_n$  where each  $b_j < a$ . Each segment  $j$  has a probability  $p(b_j) = kb_j$  intersecting one of the parallel lines. Let

$$I_j = \begin{cases} 1, & \text{if segment } j \text{ intersects the line} \\ 0, & \text{otherwise} \end{cases}$$

The total number of intersections is then

$$T = \sum_{j=1}^n I_j,$$

and the expectation of  $T$  is

---

\* It is to be noted, though, that such a method is inefficient since the rate of convergence to the true value of  $\pi$  can be shown to be  $O(n^{-1/2})$  [Mantel, N., 1953].

† See also Uspensky [1937, p. 253] and Klain & Rota [1997, p. 1]

$$\mathbf{E}T = \sum_{j=1}^n \mathbf{E}I_j = kb$$

Applying the above formula to a circle with diameter  $2a$ , viewed as a polygonal line with an arbitrarily large number of segments, we have  $T = 2$  with probability one,  $\mathbf{E}T = 2$  and  $b = \pi a$ . We can now solve for  $k = 2/(\pi a)$  and obtain

$$p(b) = kb = \frac{2b}{\pi a},$$

the same formula as before.

Although the Needle Problem seems straightforward enough and has been widely discussed in the literature, we shall see that there are still misconceptions and obscurities about it. In particular, there appears to be a widespread belief among many commentators that Buffon actually used his needle experiment to estimate the value of  $\pi$ . Others have gone even further and claimed that Buffon's actual *intent* in performing the needle experiment was to be able to estimate the value of  $\pi$ . We shall argue that there is no evidence for either of these premises. Moreover, ever since Laplace popularized the Needle Problem in the *Théorie Analytique des Probabilités* [Laplace, P.-S., 1812, p. 360], many texts of probability and statistics have described and solved it by one of the two methods described above. However, none has presented Buffon's actual solution, so far as we know. Despite the efforts of writers such as Hanks [1966] and Holgate [1981], Buffon's actual method has escaped a full and altogether transparent mathematical explanation. We shall here present Buffon's exact mathematical reasoning by solving the Needle Problem in the way Buffon intended it.

## 2 Buffon and the estimation of $\pi$

Claims that Buffon actually used the Needle Problem to estimate the value of  $\pi$  occur very frequently in textbooks, many of which are otherwise highly reliable. Thus, we can read:

...The naturalist Buffon tossed a coin 4040 times, resulting in 2048 heads and 1992 tails. He also estimated the number  $\pi$  by throwing needles on a ruled surface and recording how many times the needles crossed a line... [Grinstead, C. M. and Snell, J. L., 1997, p. 9]

...He [Buffon] reasoned, therefore, that he could experimentally determine the value of  $\pi$  by making repeated trials. [Hamming, R. W., 1974, p. 134]

Many others have asserted that Buffon actually devised the Needle Problem to be able to experimentally determine the value of  $\pi$  :

...Buffon designed the experiment in order to estimate the numerical value of  $\pi$ . [Grimmett, G. and Stirzaker, D., 2001, p. 100]

...In the late eighteenth century, Count de Buffon, French naturalist and avid needle tosser, designed this experiment to estimate a numerical value of  $\pi$ . [Olofsson, P., 2012, p. 172]

Insofar as history books on probability are concerned, none of them, including Todhunter [1865], seem to have made such claims, except Maistrov [1974, p. 120]. The latter states that Buffon made 2048 tosses to determine the value of  $\pi$ . However, the 2048 tosses (which is wrongly given in Todhunter, p. 346, as 2084 tosses) refer to an experiment Buffon alluded to in the *Essai* whereby he had a child throw coins in the air in the context of the St Petersburg Problem, not the Needle Problem.

Indeed, nowhere in his writings does Buffon refer to the estimation of  $\pi$  by using the Needle Problem, nor is there any indication that he devised the experiment in order to estimate this constant. Actually, the symbol  $\pi$  occurs nowhere in the *Essai* because the symbol was not commonly in use in France at that time [Cajori, F., 1928, p. 12; Hanks, L., 1966, p. 51].\* That Buffon never estimated  $\pi$  is perhaps the reason why all claims to the contrary have *never* given any reference to Buffon's writings where this actually occurred.

---

\* Cajori notes that the symbol  $\pi$  was first used by William Jones in 1706. In the middle of the 18<sup>th</sup> century the use of the symbol gradually spread amongst mathematicians but it was only in the latter part of that century that it gained popularity amongst French mathematicians. In any case, the symbol  $\pi$  had only been defined as the ratio of the circumference and the diameter of a circle, so it is not surprising that Buffon adopted the same approach.

Why then did Buffon devise the Needle Problem? Why did he not estimate the value of  $\pi$ ? And why it is still widely believed that Buffon used the Needle Problem to estimate the value of  $\pi$ ? For the first question, we refer to Buffon himself:

Analysis is the only instrument that has been used up-to-now in the science of probabilities to determine and to fix the ratios of chance; Geometry appeared hardly appropriate for such a delicate matter; nevertheless if one looks at it closely, it will be easily recognized that this advantage of Analysis over Geometry is quite accidental, and that chance according to whether it is modified and conditioned is in the domain of geometry as well as in that of analysis; to be assured of this, it is enough to see that games and problems of conjecture ordinarily revolve only around the ratios of discrete quantities; the human mind, rather familiar with numbers than with measurements of size, has always preferred them; games are a proof of it because their laws are a continual arithmetic; to put therefore Geometry in possession of its rights on the science of chance is only a matter of inventing some games that revolve on size and on its ratios or to analyse the small number of those of this nature that are already found...[Buffon, G.-L. L., 1777, p. 95; Hey, J. D. et al., 2010, p. 275]

Buffon's aim is thus to show that probability can be calculated not only through "analysis" using discrete quantities, as had been the practice, but also through geometry using continuous measures. This was a major breakthrough, marking the birth of geometric probability\*, and is testament to Buffon's often underrated mathematical genius. Regarding the second question, in the absence of any evidence, one can only speculate that Buffon knew that his Needle Problem would be a rather inefficient way of estimating  $\pi$  since several accurate estimates of this constant already existed at the time (not to mention Archimedes' method or the simplicity of approximating  $\pi$  by mechanical means). This would also explain why he does not even mention  $\pi$  in his derivation, holding forth instead in terms of circumferential arc lengths and areas of cycloids. The answer to the third question is not difficult. Given Buffon's predilection for the experimental method, it would seem natural that Buffon would have employed a pupil to perform the needle experiment as he did in the St Petersburg experiment.

---

\* Which later developed into integral geometry.

Indeed the suggestion that the Needle Problem could be used to estimate  $\pi$  was first made by Laplace in the *Théorie Analytique des Probabilités* [Laplace, P.-S., 1812, p. 360]. Moreover, Dorrie reports that Wolf in Zurich (1850) was the first to have actually performed the experimental estimation by throwing a needle 36 mm long between parallels separated by 45 mm a total of 5000 times [Dorrie, H., 1965, p. 77]. He found the probability of intersection to be approximately .5064 from which he estimated  $\pi$  to be 3.1596. In 1855, a certain Mr Ambrose Smith of Aberdeen made 3204 trails with a rod of three-fifths of the distance between the planks, and found  $\pi \approx 3.1553$  [De Morgan, A., 1915, p. 283]. Using ‘additional precautions’, a Captain Fox made 1,200 trials obtaining  $\pi \approx 3.1419$ . Finally, we have the case of the Italian mathematician Lazarrini: in 1901, by using 3,408 trials he obtained a value of  $\pi$  accurate up to six places of decimal to its true value. Using reasonable assumptions, Coolidge showed that the probability of attaining such accuracy by chance alone is about .014 and concluded:

It is much to be feared that in performing this experiment Lazzerini ‘watched his step’. [Coolidge, J. L., 1925, p. 82]

Indeed, as Kendall and Moran (1963, p. 71) have pointed out, many of the accurate approximations to  $\pi$  that were obtained used “optional stopping.”

### **3 Buffon’s actual derivation of the probability of intersection**

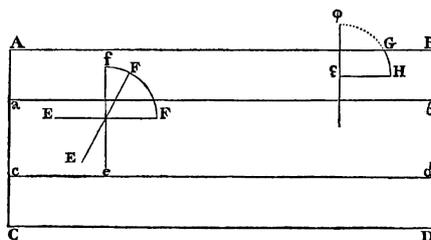
We previously mentioned that none of the subsequent mathematicians writing on the Needle Problem gave Buffon’s original derivation. The reason for this omission is simple: Buffon’s derivation is abstruse, even obscure, and may not have been completely understood. Although Hanks [1966] and Holgate [1981] attempted to delineate Buffon’s method, their efforts were not completely successful in making Buffon’s approach transparent. Writers have therefore instead preferred to give one of the two methods we provided above, whilst being full of praise regarding Buffon’s original method. Thus, we can read from one of the early major researchers in geometric probability:

The mathematical ability evinced by BUFFON may well excite surprise; that one whose life was devoted to other branches of science should have had the sagacity to discern the true mathematical principle involved in a question of so entirely novel a character, and to reduce them correctly to calculation by means of the integral calculus, thereby opening up a new region of inquiry to his successors, must move us to admiration for a mind so rarely gifted. [Crofton, M., 1868]

Similarly, without delineating the actual method used by Buffon, the great classical probabilist Károly Jordan admits:

The...problem...was solved by Buffon in a very ingenious way...[Jordan, K., 1972, p. 518]

Apart from the lack of details, what seems even more baffling in Buffon's derivation is his use of the cycloid to obtain the probability of intersection. However, let us describe Buffon's derivation before giving our explanation of his method.



**Figure 2 Buffon's diagram of the Needle Problem (taken from the *Essai d'Arithmétique Morale* [Buffon, G.-L. L., 1777, p. 102])**

Buffon considers a needle (or "baguette") EF of length  $2b$  (see left of Fig. 2) which is thrown between two parallel lines AB and CD, each of length  $f$ , and separated by a distance  $2a$ . Also the arc length  $fF$  (F lying on the horizontal line EF) is denoted by  $c$  (one quarter the circumference of a circle with radius  $b$ ). Buffon first considers the case when the centre of the needle is a distance greater than  $b$  from each of the parallel lines, i.e., within the rectangle  $abcd$ . Then the needle cannot possibly intersect either of the parallel lines AB and CD. Buffon says that this has probability

$$(\overline{a-b})f, \quad (1)$$

where  $\overline{a-b}$  is the absolute value of  $a-b$  (Buffon, however, does not make this clarification).

Next Buffon considers the case when the centre of the needle is outside the rectangle  $abcd$  (see right of Fig. 2). He denotes the arc length along the curve  $\varphi G$  by  $y$  and the infinitesimal length on  $\varepsilon\varphi$  by  $dx$ . Then Buffon writes for the second case,

$$\Pr\{\text{intersection}\} = \left(\int y dx\right) f \quad (2)$$

$$\Pr\{\text{no intersection}\} = \left(\overline{bc} - \int y dx\right) f \quad (3)$$

Combining both cases,

$$\frac{\Pr\{\text{intersection}\}}{\Pr\{\text{no intersection}\}} = \frac{\left(\int y dx\right) f}{\left(\overline{bc} - \int y dx\right) f + (a-b)cf} = \frac{\int y dx}{ac - \int y dx}. \quad (4)$$

The above implies that

$$\Pr\{\text{intersection}\} = \frac{\int y dx}{ac}.$$

Buffon now states that  $\int y dx$  is the area of part of a cycloid whose generating circle has diameter  $2b$  and that “we know that this area is”

$$\int y dx = b^2. \quad (5)$$

After this remarkable statement, Buffon is able to obtain the correct answer:

$$\Pr\{\text{intersection}\} = \frac{b^2}{a(\pi b/2)} = \frac{2b}{\pi a}.$$

Buffon’s breathtaking piece of analysis will leave many wondering where exactly Eqs (1) to (4) come from. And whence the cycloid?

Buffon’s reasoning can be understood as follows. Let us assume that  $b < a$  (as Buffon tacitly did) and let the distance between the line  $AB$  and the point  $\varepsilon$  in Fig. 2 be

denoted by  $x$ . Then, in Fig. 2, we have  $\angle \phi \varepsilon G = y/b$  and  $x = b \cos \angle \phi \varepsilon G$ . Thus, the relationship between  $x$  and  $y$  is  $y = b \arccos(x/b)$  (see Fig. 3). Note that this equation was not given by Buffon.

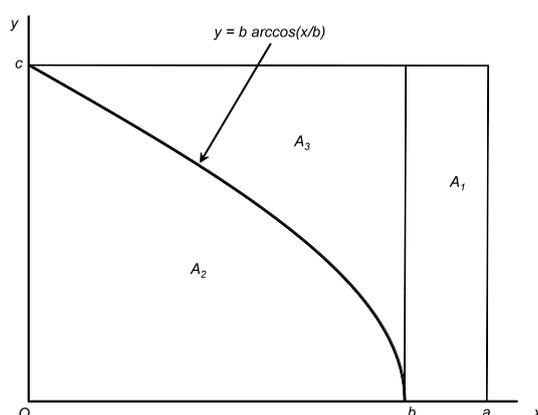


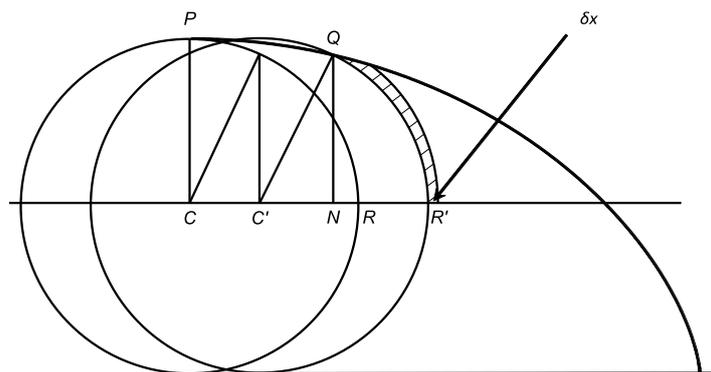
Figure 3 The favorable and unfavorable areas in Buffon's analysis

The first case considered by Buffon, namely when the centre of the needle is a distance greater than  $b$  from each of the parallel lines, is  $x > b$  and corresponds to the area  $A_1 = (a-b)c$  in Fig. 3. But Buffon is thinking in terms of volumes, so we can imagine a third axis perpendicular to  $x$  and  $y$  such that the areas  $A_1, A_2$ , and  $A_3$  are the cross-sections of a three-dimensional volume of height  $f^*$ . Therefore, multiplying by  $f$ , Buffon can then write the *total* unfavorable volume for the first case as in Eq. (1). Although Buffon called this a *probability*, he undoubtedly was referring to an unfavorable *space*, in view of the addition he does in the denominator of Eq. (4). Next the second case considered by Buffon is  $x < b$  and corresponds to the area  $A_2 + A_3$ . From Fig. 3, for this second case, the favorable area is  $A_2 = \int y dx$  and the unfavorable area is

---

\* The length  $f$  is redundant since it cancels out eventually, and Buffon could easily have done away with it. However, Buffon was no doubt thinking geometrically in terms of possible cases (i.e., points in the rectangle of length  $f$  and width  $2a$ ) in which case the arc length  $y$  varying between 0 and  $c$  provided a third dimension.

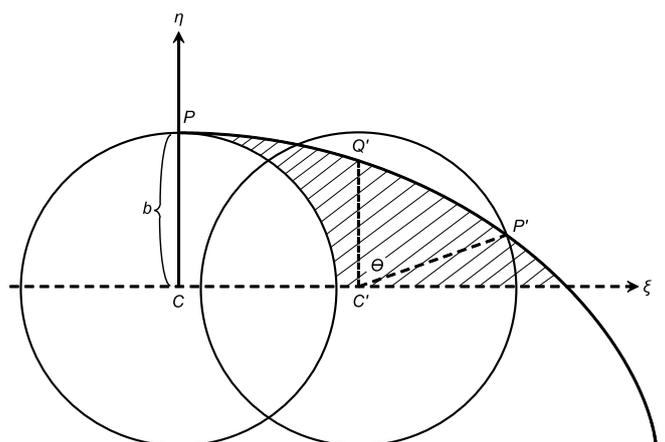
$A_3 = bc - \int y dx$ . Again, multiplying by  $f$ , these two volumes correspond to Buffon's Eq. (2) and (3), respectively. Now comes the difficult part, namely the cycloid and Eq. (5). In view of the fact that  $y = b \arccos(x/b)$ , the natural task for Buffon would have been to simply integrate it to determine  $A_2$  and hence  $A_3$ . However, Buffon does not do the integration<sup>\*</sup>, and instead resorts to an ingenious trick involving the cycloid.



**Figure 4** Holgate's explanation of the connection between the Needle Problem and the cycloid

But how did Buffon make the connection between the Needle Problem and the cycloid? An intuitive answer is provided by Holgate [1981]. Consider Fig. 4 which shows an initial point  $P$  on the generating circle (with centre  $C$ ) of the cycloid. As the circle rolls toward the right without sliding,  $P$  reaches the point  $Q$  at some later time, lying on a displaced circle with centre  $C'$ . Holgate identifies the distance  $C'N$  in Fig 4 with the distance  $x$  that the point  $\varepsilon$  is below the horizontal line  $AB$  in Fig. 2. Now  $RR' = C'N = x$ . Then Holgate reasons that as  $x$  is increased by an infinitesimal amount  $\delta x$ , the favorable area increases infinitesimally as shown in the shaded region in Fig. 4. Hence the total area is that which is shown shaded in Fig. 5 and gives the favorable space for the second case considered by Buffon. This area had previously been identified by Hanks [1966, p. 55].

<sup>\*</sup> Nor is there any evidence that he would have been able to perform such an integral.



**Figure 5** The shaded area under the cycloid is equal to the favorable area for the second case considered by **Buffon**

Would the kind of intuition used by Holgate have been enough for Buffon to state so assertively that the favorable area in the second case would be *exactly* equal to the shaded area under the cycloid shown in Fig. 5? We contend that Buffon must have used some additional reasoning, likely as follows. Consider the cycloid given in Fig. 5 with starting point  $P$ , located on the top of its generating circle. As the latter rolls to the right, suppose  $P$  is at  $P'$  at some instant, having rotated through an angle  $\theta$ . Then  $\theta = \angle P'C'Q'$  and (in modern terms of analytic geometry) the coordinates  $(\xi, \eta)$  of  $P'$  are  $\xi = b(\theta + \sin \theta)$  and  $\eta = b \cos \theta$ . Therefore, integrating the horizontal lengths with respect to  $d\eta$ , the shaded area in Fig. 5 is

$$\int_0^b \left\{ \xi - \sqrt{b^2 - \eta^2} \right\} d\eta.$$

Now  $\cos \theta = \eta/b$ , so that  $\xi = b(\theta + \sin \theta) = b \arccos(\eta/b) + \sqrt{b^2 - \eta^2}$ , implying that the shaded area is  $\int_0^b b \arccos(\eta/b) d\eta$ . But this is precisely the area  $A_2 = \int_0^b b \arccos(x/b) dx$  (see Fig. 3) that Buffon sought. Notice that we have not done any actual integration in the above, and such geometric reasoning would have certainly been within Buffon's mathematical abilities.

Finally, Buffon seems to be quite confident too that the shaded area in Fig. 5 should be equal to  $b^2$ . Though it is rather unfamiliar nowadays, this fact must have been known to him, as Holgate [1981] effectively argues. That the shaded area equals  $b^2$  is a consequence of the fact that, in Fig. 6, the area of the rectangle AB is equal to the sum of the areas of the rectangle BC and the shaded section of the cycloid. A proof of this classical theorem can be found in Proctor [1878, p. 10].

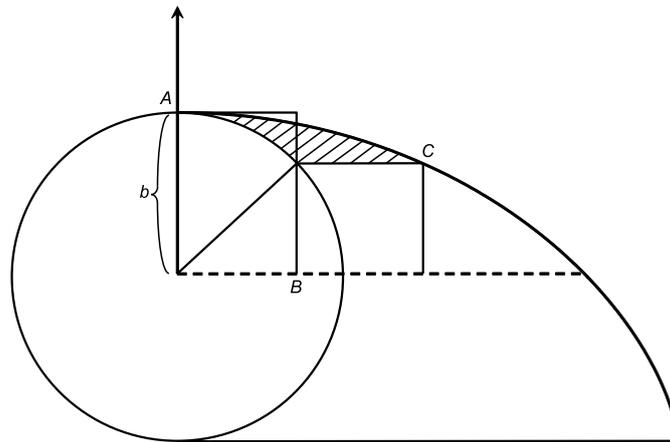


Figure 6 The justification for why the shaded area in Fig. 5 is equal to  $b^2$ .

We previously mentioned the important works of Hanks [1966] and Holgate [1981] in deciphering Buffon's reasoning, and stated that their efforts were not entirely successful. Thus, Hanks treatment of Buffon's solution raises more questions than it answers, in particular with the latter's use of Eqs (1)-(5). However, Hanks is successful in correctly identifying the shaded area in Fig. 5 as being the one that Buffon refers to in his solution. Similarly, Holgate provides no clear explanation of the exact steps in Buffon's reasoning. He instead provides intuitive reasoning as to how Buffon might have made the link between the Needle Problem and the cycloid, and then concludes:

My conjecture is that Buffon used some such reasoning as this to obtain the number of favourable cases..."

## 4 Conclusion

We have sought to achieve two aims in this paper. First, contrary to many statements that continue to be made regarding Buffon's estimation of  $\pi$  in the Needle Problem, we have argued there is no evidence within Buffon's writings that any such attempt was made. The fact that Buffon did not estimate  $\pi$  surely does not change anything insofar as the theory of probability is concerned, except perhaps to correct an historical inaccuracy. Secondly, we have provided details of what we believe were Buffon's actual geometric reasoning in solving the Needle Problem. Although Buffon has often been described as mathematically unsophisticated, a detailed analysis of his resolution of the Needle Problem reveals a very capable analytical mind.

Our contention that Buffon never estimated  $\pi$  using his Needle Problem may seem overreaching insofar as it is difficult to prove an assertion of non-existence. Nevertheless, the likelihood ratio in favor of our "null" hypothesis is currently overwhelming (much writing but no evidence). Yet the balance can be easily reversed by a single legitimate citation. We welcome any such proof of existence.

## References

1. Barbier M.E., 1860 Note sur le problème de l'aiguille et le jeu du joint couvert. *J Math Pures Appl* 5: 272-286.
2. Bernoulli D., 1738 Specimen theoriae novae de mensura sortis. *Commentarii Academiae Scientiarum Imperialis Petropolitanae* V 175-192 (translated and republished as "Exposition of a new theory on the measurement of risk," *Econometrica* 22 (1954): 23–36).
3. Buffon G.-L.L.: La Méthode des Fluxions et des Suites Infinis (de Newton). Paris, Debure l'Aîné, 1740.
4. Buffon G.-L.L.: Histoire Naturelle Générale et Particulière, avec la Description du Cabinet du Roy (Vol 1). Paris, 1749.
5. Buffon G.-L.L., 1777 Essai d'Arithmétique Morale. *Supplement a l'Histoire Naturelle* 4: 46-148.

6. Cajori F.: A History of Mathematical Notations (Vol. 2). Chicago, Open Court, 1928.
7. Coolidge J.L.: An Introduction to Mathematical Probability. Oxford, Oxford University Press, 1925.
8. Crofton M., 1868 On the theory of local probability. *Phil Trans R Soc Lond* **158**: 181-199.
9. De Morgan A.: A Budget of Paradoxes. ed 2nd, Chicago, Open Court Publishing Co., 1915.
10. Dorrie H.: 100 Great Problems of Elementary Mathematics: Their History and Solutions. New York (Originally published in German under the title *Triumph der Mathematik*, Physica-Verlag, Würzburg, 1958), Dover, 1965.
11. Gorroochurn P.: Classic Problems of Probability. NJ, Wiley, 2012.
12. Gouraud C.: Histoire du Calcul des Probabilités Depuis ses Origines Jusqu'a nos Jours. Paris, Librairie D'Auguste Durand, 1848.
13. Gridgeman N.T., 1960 Geometric probability and the number  $\pi$ . *Scripta Mathematica* **25**: 183-195.
14. Grimmett G., Stirzaker D.: Probability and Random Processes. ed 3rd, Oxford, Oxford University Press, 2001.
15. Grinstead C.M., Snell J.L.: Introduction to Probability. ed 2nd, American Mathematical Society, 1997.
16. Hamming R.W.: Numerical Methods for Scientists and Engineers. ed 2nd, New York, McGraw-Hill, 1974.
17. Hanks L.: Buffon Avant l'Histoire Naturelle. Paris, Presses Univ. France, 1966.
18. Hey JD, Neugebauer M, Pasca CM: Georges-Louis Leclerc de Buffon's 'Essays on Moral Arithmetic'; in Ockenfels A, Sadrieh A, (eds): The Selten School of Behavioral Economics. Springer, 2010.
19. Holgate P., 1981 Studies in the History of Probability and Statistics XXXIX. Buffon's Cycloid. *Biometrika* **68**: 712-716.
20. Jordan K.: Chapters in the Classical Calculus of Probability. Budapest, Hungary, Akadémiaio Kiadó, 1972.

21. Klain D.A., Rota G.-C.: Introduction to Geometric Probability. Cambridge University Press, 1997.
22. Laplace P.-S.: Théorie Analytique des Probabilités. Paris, Mme Ve Courcier, 1812.
23. Locke J.: An Essay Concerning Human Understanding. London, 1690.
24. Maistrov L.E.: Probability Theory: A Historical Sketch. New York, Academic Press, 1974.
25. Mantel N., 1953 An Extension of the Buffon Needle Problem. *The Annals of Mathematical Statistics* **24**: 674-677.
26. O'Beirne T.H.: Puzzles and Paradoxes. Oxford, Oxford University Press, 1965.
27. Olofsson P.: Probability, Statistics, and Stochastic Processes. ed 2nd, NJ, John Wiley & Sons, 2012.
28. Proctor R.A.: A Treatise on the Cycloid and all Forms of the Cycloidal Curves. London, Longmans, Green, and Co., 1878.
29. Todhunter I.: A History of the Mathematical Theory of Probability From the Time of Pascal to That Laplace. London (Reprinted by Chelsea, New York, 1949,1965)., Macmillan, 1865.
30. Uspensky J.V.: Introduction to Mathematical Probability. New York, McGraw-Hill, 1937.
31. Weil F., 1961 La correspondance Buffon-Cramer. *Rev Hist Sci* **14**: 97-136.